













# An Introduction to LINEAR ALGEBRA

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**To Our Parents**



# Contents

<b>PREFACE</b>	<b>ix</b>
<b>1 SETS AND FUNCTIONS</b>	<b>1</b>
1.1 Sets	1
1.2 Operations on Sets	5
1.3 Relations	8
1.4 Functions	10
1.5 Binary Operations	17
1.6 Algebraic Structures	20
1.7 Operations on Functions	24
<b>2 VECTORS</b>	<b>33</b>
2.1 Space Coordinates	33
2.2 Vectors—Addition and Scalar Multiplication	37
2.3 Dot Product of Vectors	48
2.4 Applications to Geometry	55
<b>3 VECTOR SPACES</b>	<b>63</b>
3.1 Vector Spaces	63
3.2 Subspaces	69
3.3 Span of a Set	74
3.4 More about Subspaces	77
3.5 Linear Dependence, Independence	85
3.6 Dimension and Basis	95
<b>4 LINEAR TRANSFORMATIONS</b>	<b>107</b>
4.1 Definition and Examples	107
4.2 Range and Kernel of a Linear Map	114
4.3 Rank and Nullity	118
4.4 Inverse of a Linear Transformation	122
4.5 Consequences of Rank-Nullity Theorem	127
4.6 The Space $L(U, V)$	129
4.7 Composition of Linear Maps	133
4.8 Operator Equations	138
4.9 Applications to the Theory of Ordinary Linear Differential Equations	141

<b>5</b>	<b>MATRICES</b>	<b>148</b>
5.1	Matrix Associated with a Linear Map	148
5.2	Linear Map Associated with a Matrix	154
5.3	Linear Operations in $M_{m,n}$	158
5.4	Matrix Multiplication	163
5.5	Rank and Nullity of a Matrix	174
5.6	Transpose of a Matrix and Special Types of Matrices	180
5.7	Elementary Row Operations	185
5.8	Systems of Linear Equations	195
5.9	Matrix Inversion	200
<b>6</b>	<b>DETERMINANTS</b>	<b>203</b>
6.1	Definition	203
6.2	Fundamental Properties of Determinants	207
6.3	Proofs of Theorems	212
6.4	Cofactors	216
6.5	Determinant Minors and Rank of a Matrix	224
6.6	Product of Determinants	227
6.7	Applications to Linear Equations	230
6.8	Eigenvalues, Eigenvectors	233
6.9	Wronskians	238
6.10	Cross Product in $V_3$	241
<b>7</b>	<b>MORE MATRIX THEORY</b>	<b>250</b>
7.1	Similarity of Matrices	250
7.2	Inner Product Spaces	258
7.3	Orthogonal and Unitary Matrices	264
7.4	Application to Reduction of Quadrics	266
	<b>APPENDIX Ordinary Linear Differential Equations</b>	<b>276</b>
	<b>ANSWERS TO PROBLEMS</b>	<b>297</b>
	<b>INDEX</b>	<b>325</b>

# Preface

In Indian universities the emphasis at the undergraduate level has been much more on calculus than on linear algebra and matrix theory. Today, however, the need for linear algebra and matrix theory as an essential part of undergraduate mathematics is recognised. The Bi-national Conference in Mathematics Education and Research (June 1973) recommended Elementary Linear Algebra as a compulsory course for all students at the undergraduate level. It has since been generally agreed that before a student begins to specialise in the discipline of his choice—whether it be mathematics, science, engineering, social science, or management—he must be exposed at least once to both calculus and linear algebra; such an exposure will familiarise him with the concepts and techniques of continuous mathematics (calculus), and the concepts, methods, and logic of modern discrete mathematics (linear algebra).

This book is the outcome of a planned effort to teach linear algebra as a second course in the mathematics curriculum introduced at the undergraduate level several years ago at Birla Institute of Technology and Science (BITS), Pilani. The students who take this course have had a semester of elementary calculus and analytical geometry. However, a knowledge of the fundamental properties of continuous and differentiable functions in terms of their addition, scalar multiplication, and multiplication is sufficient for an understanding of this volume.

The treatment throughout is rigorous yet lucid. The fact that the majority of students who would use this text may not ultimately become mathematicians or physicists has not inhibited our development of the subject. We strongly believe that present-day users of mathematics, instead of being content with a hybrid of mathematical tools and gymnastics which merely graze the subject, should delve deep by training in concrete matter-of-fact arguments. The conceptual framework of linear algebra and matrix theory provides the most efficient means for this training; for, in one sense, matrices and linear equations form a concrete foundation, and vector spaces and linear transformations give the flavour of the abstract grandeur of modern mathematics. At the same time, as the freshmen we are addressing may not have had any grounding in abstract



mathematics, we have made a special effort to smoothen their first encounter with methods of proof.

Theorems are proved in full (the end of proof is indicated by ■), except in rare cases where they are beyond the scope of the book. In these instances the student is suitably instructed. Where certain consequences of earlier results are stated as **FACTS**, the student will find he has been sufficiently equipped in advance to prove them himself. The large number of worked-out examples which have been woven into the text help the student to move back and forth from the concrete to the abstract. The sets of problems—numerical, objective, and theoretical—interpolated at the end of almost every article are a drill on the text. Answers to the numerical problems appear at the end of the book; the objective questions which are of the “true-false” type are intended to help the student in a self-assessment of his conceptual understanding of the subject under study.

Chapter 1 deals with sets and functions and gradually introduces the language of modern mathematics. A teacher may adjust his pace in this chapter to suit the standard of his class. Algebraic structures, such as groups, rings, and fields, have been discussed only to the extent needed.

Chapter 2 provides the concrete geometric structure of 2- and 3-dimensional vector spaces. It leads the student to the problems of geometry through vectors and prepares the ground for Chapter 3 which gets into the essence of the subject. Here the theory of vector spaces, and the concepts of linear dependence and linear independence, dimension and basis are treated elaborately. Though infinite-dimensional vector spaces are also considered, the emphasis throughout is on finite-dimensional vector spaces.

Chapter 4 aims to familiarise the student with the fundamental properties of linear transformations. The rank-nullity theorem and its consequences are presented in detail. The theory developed so far is applied to operator equations and, in particular, to differential equations. This application discloses that the solution space of the  $n$ -th order normal homogeneous linear differential equation is an  $n$ -dimensional subspace of the space of  $n$ -times continuously differentiable functions. The further application to the theory of ordinary linear differential equations is detailed in the Appendix. However, we have not attempted to make the treatment of differential equations self-contained.

The elaborate build-up on vector spaces and linear transformations begins to pay dividends in Chapter 5 which starts by establishing the link between linear transformations and matrices. In our experience, the welter of mathematical detail on matrices in this chapter is easily assimilated by the student because of the knowledge of linear transformations he has gained in Chapter 4. He is thrilled to see that the elementary

(apparently trivial) row operations on matrices finally result in the solution of linear equations in all their ramifications. Naturally, the chapter ends with matrix inversion.

Now the student is ready for determinants, presented in Chapter 6. When he comes to determinant minors and the rank of a matrix he realises the importance of the emphasis in Chapter 3 on the concept of linear dependence and linear independence. The theorem giving the connection between the rank of a matrix (already defined in Chapter 5 by means of independence concepts) and the order of its nonzero minors is the crux of the content here. The ease with which it is proved justifies the efforts taken in the development of the subject in earlier chapters. Applications to linear equations, and a brief account of eigenvalues and eigenvectors of matrices, Wronskians, and the cross-product in  $V_3$  give an idea of what determinants can do.

Chapter 7 gives a glimpse of the theory of orthogonal and unitary matrices, similarity transformations and their application to the geometry of quadrics. When the student reaches this chapter, he easily recognises the connection between linear algebra and geometry.

The student should guard against conceptual errors of three types : 'finite dimension' versus 'infinite dimension'; 'real scalar' versus 'complex scalar'; and 'non-empty set' versus 'empty set'. When in doubt regarding hypotheses he should invoke what may be called an 'emergency axiom' : The suitable alternative in each relevant pair(s) is included as an additional hypothesis.

The text can be adapted to suit different curricula : as a one-year course running three hours a week; as a one-semester course running five or six hours a week (as is the practice at BITS); or, by a judicious selection of topics, as a onesemester course running three hours a week. It can be used during any year at the undergraduate level or as part of a first course in linear algebra at the postgraduate level, if no matrix theory has been done till then. However, as the topics have been arranged sequentially, any student who wishes to change the order of topics will need guidance.

We wish to thank Dr. C. R. Mitra (Director, BITS) and others at BITS who encouraged our efforts in writing this book under the Course Development Scheme and provided all the necessary assistance in duplicating and class-testing its earlier versions over a period of three years. To the BITS students—about 1200—of these three years, we are also indebted for their lively response to our experiments in pedagogy.

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**xii / PREFACE**

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**Our last words of affectionate gratitude are reserved for our wives and family members who continually cheered our years of effort at the project.**

**We shall welcome all suggestions for improvement of the book.**

**Pilani  
September 1938**

**V. Krishnamurthy  
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# Sets and Functions

We shall begin with a discussion on 'sets'. Immediately, we get into the first requirement of mathematics, viz. proper definitions for technical terms. A major concern of mathematics is precision not only in the calculations of quantitative information, but also in the communication of thought. This is why importance is given to definitions in mathematics. In this book we shall come across a large number of definitions. Each definition will introduce a new technical term or a new concept in terms of preceding definitions. The very comprehension of a definition may often depend on the logical development of the subject up to that point.

However, the first technical term, namely, 'set' will be introduced without a precise definition. The reason is obvious : The moment we attempt to define 'set', we get into words such as 'collection' or 'aggregate', which themselves need to be defined. We have to draw the line somewhere !

## 1.1 SETS

The meaning of 'set' is intuitively understood as *a well-determined collection of objects*, called its 'members' or 'elements'. The objects (members or elements) are said to 'belong to' the set or to be 'in' the set. Here all the words in quotation marks are taken to be undefined terms. To illustrate the meaning of 'set', let us consider some examples.

**Example 1.1** The collection of the three boys : Arun, Mohan, and Ram.

**Example 1.2** The collection of the three symbols :  $\Delta$ ,  $\square$  and  $\Omega$ .

**Example 1.3** The collection  $N$  of all natural numbers.

**Example 1.4** The collection  $Z$  of all integers.

**Example 1.5** The collection  $Q$  of all rational numbers.

**Example 1.6** The collection  $R$  of all real numbers.

**Example 1.7** The collection of all the past presidents of India.

**Example 1.8** The collection of all the first year students of Birla Institute of Technology and Science (BITS).

## 2 | SETS AND FUNCTIONS

*Example 1.9* The aggregate of the living men in the world whose height exceeds 2 metres.

*Example 1.10* The aggregate of the roots of the equation  $x^{37} - 1 = 0$ .

*Example 1.11* The aggregate of the integers that leave a remainder 2 when divided by 5.

*Example 1.12* The group of cricketers who were out for 99 runs in a test match.

*Example 1.13* The collection of all positive primes.

*Example 1.14* The collection of derivatives of all orders of the function  $e^x$ .

*Example 1.15* The collection  $C$  of all complex numbers.

All these are examples of sets. We can construct several such examples. Let us now consider two collections which are not sets :

- (i) The collection of some natural numbers.
- (ii) The collection of the politicians of India.

In (i) it is not clear which numbers are included in the collection. Whether the number 2 is in the collection or not cannot be answered without first explaining the word 'some'. Again, in (ii) the question whether a specific person is a politician or not would get different responses from different persons. Thus, collections (i) and (ii) are not 'well determined'.

In contrast to these two examples it is worthwhile to analyse Examples 1.12 and 1.13. In Example 1.12 we note that every cricketer is either in the group or he is not in the group. We do not have to check any records to say this. In Example 1.13, again, either a number is a prime or it is not a prime. Here it matters little whether it is known that a particular number is a prime or not. In other words, it is immaterial whether we can answer the question : Is this particular object in the given collection or not ? What matters for specifying a set is to know unambiguously that only one of the two answers is possible : The object in question belongs to the given collection or it does not belong to it.

Thus, we can elaborately describe a set as a collection of objects which is well determined in the sense that, for every object, there should be only two possibilities available unambiguously : either it belongs or it does not belong to the collection.

If  $A$  is a set and  $x$  is an element of  $A$ , then we write  $x \in A$  ( $\in$  is read as 'belongs to' or 'is in' or 'is an element of' or 'is a member of'). The negation of this is denoted by  $x \notin A$  ( $\notin$  is read as 'does not belong to' or 'is not in' or 'is not an element of' or 'is not a member of').

Instead of such a detailed description of sets, two kinds of symbolic descriptions are very often used. One is by listing, if possible, all the

elements of the set within braces, e.g. sets of Examples 1.1 and 1.2 are respectively written as

$$\{\text{Arun, Mohan, Ram}\}$$

and

$$\{\triangle, \square, \Omega\}.$$

The other is by using a dummy element, say  $x$ , and writing the characteristic properties of  $x$  which precisely make it an element of the set. Thus, the set of elements characterised by the properties, say  $P, Q, \dots$ , is written as

$$\{x \mid x \text{ satisfies } P, Q, \dots\}$$

or

$$\{x : x \text{ satisfies } P, Q, \dots\}.$$

(‘ $\mid$ ’ and ‘ $:$ ’ are read as ‘such that’.) Thus, the sets in Examples 1.11 and 1.12 may respectively be written as

$$\{x \mid x \text{ is an integer and } x = 5k + 2 \text{ for some integer } k\}$$

and

$$\left\{ x : \begin{array}{l} x \text{ is a cricketer who was out for just 99 runs} \\ \text{in a test match} \end{array} \right\}.$$

Note that in the first method the order in which the elements are listed is immaterial. Thus,  $\{\text{Arun, Mohan, Ram}\}$  and  $\{\text{Mohan, Arun, Ram}\}$  are the same sets. We shall make this precise by defining the equality of sets.

**1.1.1 Definition** Two sets  $A$  and  $B$  are said to be *equal* if every member of  $A$  is a member of  $B$ , and every member of  $B$  is a member of  $A$ . In such a case we write  $A = B$ .

For example,  $\{0, 1, 2, 3\} = \{2, 1, 0, 3\}$ . Also  $Z = \{x \mid x \text{ is an integer}\} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ .

**1.1.2 Convention** All our definitions are ‘if and only if’ (*iff*) statements.

For example, if a definition reads ‘A triangle is said to be equilateral *iff* all its sides are equal’, we actually mean that ‘A triangle is said to be equilateral *iff* all its sides are equal’.

In view of this convention, Definition 1.1.1 means that two sets  $A$  and  $B$  are said to be equal *iff* every member of  $A$  is a member of  $B$ , and every member of  $B$  is a member of  $A$ .

## SUBSETS

Let  $A$  and  $B$  be two sets such that every member of  $A$  is also a member of  $B$ . Then  $A$  is said to be a *subset* of  $B$ . In symbols we write  $A \subset B$  (‘ $\subset$ ’ is read as ‘is a subset of’ or ‘is contained in’). Whenever  $A$  is a subset of  $B$ , we also say that  $B$  is a *superset* of  $A$ . In symbols we write  $B \supset A$  (‘ $\supset$ ’ is read as ‘is a superset of’ or ‘contains’).

Obviously, every set is a subset (superset) of itself.

**Exercise** Prove that  $A = B$  iff  $A \subset B$  and  $B \subset A$ .

This exercise is immediate and the reader can prove it himself. In practice, whenever we want to prove that two sets  $A$  and  $B$  are equal, we prove both the inclusions, i.e.  $A \subset B$  and  $B \subset A$ .

## EMPTY SET

Consider the set of women presidents of India elected before 31st December 1974. There was no such woman president. So this set has no members in it. Such a set, i.e. a set containing no elements in it, is called an *empty set* or a *null set*. It is denoted by  $\phi$ .

It may be noted here that there is only one empty set. As a clarification of this, note that the set  $\{x : x \text{ is a real number satisfying } x^2 + 1 = 0\}$  is also empty and we can write

The set of all women presidents of India elected before

31st December 1974  $= \phi$

$$= \{x : x \text{ is a real number satisfying } x^2 + 1 = 0\}.$$

The set  $\{x : x \text{ is a real number satisfying } x^2 + x = 0\}$  consists of only one member, namely, zero. So this set can be written as  $\{0\}$ . Note that this is *not* the empty set.

Nonempty sets that consist of only a single member are called *singletons*. The set in Example 1.14 is a singleton as it consists of only one element  $e^a$ . It would be interesting (at least for cricket fans) to find out whether the set in Example 1.12 is empty or a singleton or has more members than one.

Finally, we note that  $\phi \subset A$  for all sets  $A$ . Thus, given a set  $A$ , it has two extreme subsets. One is  $A$  itself and the other is  $\phi$ . Any subset of  $A$  other than  $A$  and  $\phi$  is called a *proper subset* of  $A$ .

## Problem Set 1.1

- Let  $A$ ,  $B$ , and  $C$  be three sets such that  $A \subset B$  and  $B \subset C$ . Then prove that  $A \subset C$ .
- Let  $S_1, S_2, \dots, S_n$  be  $n$  sets such that  $S_1 \subset S_2 \subset \dots \subset S_n$  and  $S_n \subset S_1$ . Then prove that  $S_1 = S_2 = \dots = S_n$ .
- Prove the exercise given on page 3.
- Determine the set of real numbers  $x$  satisfying the following :
 

(a)  $x^2 < 1$

(b)  $x(x + 1) \leq 0$

(c)  $\frac{x+1}{x-2} \leq 0$

(d)  $x^2 - 4 \geq 0$

(e)  $(x^2 + 2)(x^2 - 1) \geq 0$

(f)  $x + 1 \leq x$
- Determine the set of complex numbers  $z$  satisfying the following :

$$(a) |z| < 4 \quad (b) \left| \frac{z+1}{z-1} \right| = \frac{1}{2} \quad (c) \left| \frac{z+i}{z-i} \right| \leq 1$$

- (d)  $z\bar{z} + 2z + 2z - 5 = 0$       (e)  $z\bar{z} + 2z + 2\bar{z} + 5 = 0$ .
6. Determine all the subsets of the set :  
 (a)  $\{0, 1, 2\}$       (b)  $\{\alpha, \beta, \gamma, \delta\}$ .
7. Let  $X$  be a set containing  $n$  elements. What is the total number of distinct subsets of  $X$  ?
8. Determine whether each of the following statements is true or false ( $N, Z, Q$ , and  $R$  are defined in Examples 1.3, 1.4, 1.5, and 1.6, respectively) :  
 (a)  $N \subset Q$ .      (b)  $\{N, Q, Z\} \subset R$ .      (c)  $N \subset \{N\}$ .  
 (d)  $N \in \{N, Q, Z\}$       (e)  $R \subset \{N, Q, R\}$ .
9. Given  $A = \{0, 1, 2, 3, 4\}$ ,  $B = \{x \in N \mid x \leq 10\}$ ,  $C = \{x \in R \mid x \leq 10\}$ , determine whether each of the following statements is true or false :  
 (a)  $A \subset B$ .      (b)  $B \subset C$ .      (c)  $C \subset A$ .  
 (d)  $1 \notin B$ .      (e)  $-1 \in B$ .      (f)  $-1 \in C$ .  
 (g)  $3 \in B$ .      (h)  $C \subset B$ .      (i)  $B = C$ .

## 1.2 OPERATIONS ON SETS

The standard operations on sets, which yield new sets from old ones, are (i) union, (ii) intersection, and (iii) complementation.

Given two sets  $A$  and  $B$ , the *union* of  $A$  and  $B$ , written as  $A \cup B$ , is defined as

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}.$$

Here ' $x \in A$  or  $x \in B$ ' means  $x$  is in  $A$  or in  $B$  or in both. Throughout this book the word 'or' will be used in the inclusive sense as here, except when otherwise stated.

The *intersection* of two sets  $A$  and  $B$ , written as  $A \cap B$ , is defined as

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}.$$

The *complement* of  $B$  in  $A$ , written as  $A \setminus B$ , is defined as

$$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}.$$

$A \setminus B$  is also denoted by  $C_A(B)$ .

**Example 1.16** Let  $A = \{1, 2, 3, 4, 5, 6\}$  and  $B = \{0, -1, 2, -3, 4, -5, 6\}$ . Then

$$A \cup B = \{-5, -3, -1, 0, 1, 2, 3, 4, 5, 6\}$$

$$A \cap B = \{2, 4, 6\}$$

$$A \setminus B = \{1, 3, 5\} \text{ and } B \setminus A = \{0, -1, -3, -5\}.$$

The geometrical illustrations in Figure 1.1, called Venn diagrams, are helpful in understanding these operations.



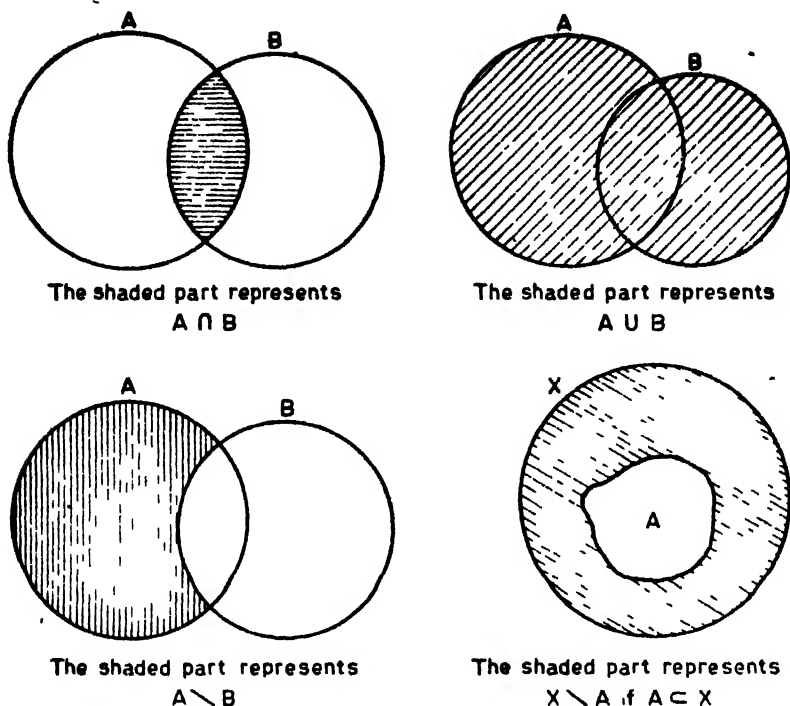


FIGURE 1.1

## CARTESIAN PRODUCT

Let  $A$  and  $B$  be two sets. Consider the set of all ordered pairs  $(x, y)$ , where  $x \in A$  and  $y \in B$ . This set is called the *cartesian product* of sets  $A$  and  $B$ , written as  $A \times B$ . In symbols we write

$$A \times B = \{(x, y) \mid x \in A, y \in B\}.$$

**Example 1.17** Let  $A = \{1, 2\}$  and  $B = \{x, y, z\}$ . Then

$$A \times B = \{(1, x), (2, x), (1, y), (2, y), (1, z), (2, z)\}.$$

**Example 1.18** Let  $R$  be the set of real numbers. Then  $R \times R$  (also written as  $R^2$ ) is the set of all ordered pairs  $(x, y)$ , where  $x$  and  $y$  are real numbers, that is,

$$R \times R = \{(x, y) \mid x, y \text{ are real numbers}\}.$$

Geometrically,  $R$  represents the set of points of a straight line (called the real line), and  $R \times R$  represents the set of points of a plane.

### Problem Set 1.2

1. Let  $A$ ,  $B$ , and  $C$  be the sets as defined in Problem 9 (Problem Set 1.1). Determine

- (a)  $A \cup B$  (b)  $A \cap B$  (c)  $A \cap C$   
 (d)  $B \cup C$  (e)  $A \setminus B$  (f)  $B \setminus C$   
 (g)  $C \setminus A$  (h)  $A \times B$

2. Let  $A = \{0, 1, 2, 3, 4\}$ ,  $B = \{x \in N \mid x < 20\}$ ,  
 $C = \{x \in N \mid x < 3\}$ ,  $D = \{x \in N \mid x \text{ is divisible by } 7\}$ .

Determine

- (a)  $A \cup B$  (b)  $A \cup D$  (c)  $B \cap C$   
 (d)  $B \cap D$  (e)  $A \cap C$  (f)  $A \cap D$   
 (g)  $A \setminus B$  (h)  $A \setminus D$  (i)  $B \setminus C$   
 (j)  $B \setminus D$  (k)  $C \setminus D$  (l)  $A \times B$

3. If the sets  $A, B, C$ , and  $D$  are as defined in Problem 2, then determine whether each of the following statements is true or false:

- (a)  $B \cap C \subseteq D$  (b)  $A \subseteq B$  (c)  $A \subseteq C$   
 (d)  $A \cap D = \emptyset$  (e)  $A \cap C \subseteq B$  (f)  $C \setminus A \subseteq D$

4. Let  $A = \{a, \beta, \gamma\}$ ,  $B = \{\beta, \alpha, \theta\}$ ,  $C = \{a, \gamma, \epsilon\}$ . Determine

- (a)  $A \times B$  (b)  $A \times C$  (c)  $B \times C$

5. Describe the following subsets of  $R$

- (a)  $\{x \mid x > -7\} \cup \{x \mid x < 0\}$  (b)  $\{x \mid x > -1\} \cup \{x \mid x < 1\}$   
 (c)  $\{x \mid x = -1\} \cup \{x \mid x = 1\}$  (d)  $\{x \mid x \geq -1\} \cap \{x \mid x < 1\}$   
 (e)  $\{x \mid x < -1\} \cap \{x \mid x > 1\}$  (f)  $\{x \mid x > -1\} \cap \{x \mid x \geq 0\}$

6. If  $A, B, C$ , and  $D$  are any sets, prove that

- (a)  $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$   
 (b)  $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$

7. Give an example in which

$$(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$$

8. If  $A, B$ , and  $C$  are any sets, prove that

- (a)  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$   
 (b)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$   
 (c)  $A \cup (B \cup C) = (A \cup B) \cup C$   
 (d)  $A \cap (B \cap C) = (A \cap B) \cap C$

9. Prove *DeMorgan's Theorem* :

If  $A$  and  $B$  are both subsets of a set  $S$ , then

- (a)  $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$   
 (b)  $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$

10. True or false ?

- (a) If  $A$  and  $B$  are two sets,  $A \cap B$  is the largest subset of  $A \cup B$ , which is contained in both  $A$  and  $B$ .  
 (b) If  $A$  and  $B$  are two sets,  $A \cup B$  is the smallest superset of  $A \cap B$ , which contains both  $A$  and  $B$ .

- (c) If  $A$  is nonempty,  $A \setminus B$  can never be empty.
- (d) The union of two intervals in  $R$  is an interval.
- (e)  $A \times B$  is not equal to  $B \times A$ .
- (f) This text has not yet given the definition of an infinite set.
- (g) This text has not yet considered an infinite set.
- (h)  $R \times R$  is the set  $C$  of complex numbers.

## 1.3 RELATIONS

Let  $A$  be a nonempty set. A subset  $\mathcal{R}$  of  $A \times A$  is called a *relation* on  $A$ . If  $(x, y) \in \mathcal{R}$ , we say  $x$  is related to  $y$  by the relation  $\mathcal{R}$  and symbolically we write  $x \mathcal{R} y$ . For example,  $\mathcal{R} = \{(1, 2), (1, 3), (2, 3)\}$  defines a relation on the set  $A = \{1, 2, 3\}$ . Here  $1 \mathcal{R} 2$ ,  $1 \mathcal{R} 3$ ,  $2 \mathcal{R} 3$ . Obviously, it is the usual relation ' $<$ ', because  $1 < 2$ ,  $1 < 3$ ,  $2 < 3$ . On the same set  $A$ , the relation ' $\leq$ ' is described by the set  $\{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$ .

We note that ' $<$ ', ' $=$ ', ' $>$ ', ' $\leq$ ', etc., are relations on  $R$ ,  $N$ ,  $Z$ , and  $Q$ . 'Is the mother of', 'is the brother of', 'is married to' are relations on the set of all human beings.

**1.3.1 Definition** Let  $\mathcal{R}$  be a relation on a set  $A$ .

- (a) If  $x \mathcal{R} x$  (i.e.  $(x, x) \in \mathcal{R}$ ), for every  $x \in A$ ,  $\mathcal{R}$  is said to be a *reflexive relation*.
- (b) If, whenever  $x \mathcal{R} y$ , it is also true that  $y \mathcal{R} x$  (i.e. whenever  $(x, y) \in \mathcal{R}$ ,  $(y, x)$  also belongs to  $\mathcal{R}$ ),  $\mathcal{R}$  is said to be a *symmetric relation*.
- (c) If, whenever  $x \mathcal{R} y$  and  $y \mathcal{R} z$ , it is also true that  $x \mathcal{R} z$  (i.e. if  $(x, y) \in \mathcal{R}$  and  $(y, z) \in \mathcal{R}$ , then  $(x, z) \in \mathcal{R}$ ),  $\mathcal{R}$  is said to be a *transitive relation*.
- (d) A relation  $\mathcal{R}$  on  $A$  that is reflexive, symmetric, and transitive is called an *equivalence relation* on  $A$ .

We shall consider several examples of relations.

**Example 1.19** Let  $A = Z$ , the set of all integers. Consider the subset  $\mathcal{R}$  of  $Z \times Z$  defined by

$$\mathcal{R} = \{(x, y) : x - y \text{ is divisible by } 3\}.$$

Here  $x \mathcal{R} y$  iff  $(x - y)$  is divisible by 3.

- (a)  $\mathcal{R}$  is reflexive, because  $x - y$  is divisible by 3 for every integer  $x$ .
- (b)  $x - y$  is divisible by 3 clearly means  $y - x$  is also divisible by 3.

Hence,  $x \mathcal{R} y$  implies  $y \mathcal{R} x$ . So  $\mathcal{R}$  is symmetric.

(c) If  $x - y$  is divisible by 3 and  $y - z$  is divisible by 3, it is certainly true that  $x - z = (x - y) + (y - z)$  is also divisible by 3. Thus,  $x \mathcal{R} y$  and  $y \mathcal{R} z$  imply  $x \mathcal{R} z$ . Hence,  $\mathcal{R}$  is transitive.

Therefore,  $\mathfrak{R}$  is an equivalence relation.

**Example 1.20** Let  $A = \mathbb{R}$ , the set of real numbers. It is obvious that ' $=$ ' is an equivalence relation.

**Example 1.21** Define the relation  $\mathfrak{R}$  on  $\mathbb{Z} \setminus \{1\}$  by saying  $x \mathfrak{R} y$  iff  $x$  and  $y$  have a common factor other than 1. This relation is reflexive and symmetric, but not transitive.

**Example 1.22** On  $\mathbb{Z}$ , define  $x \mathfrak{R} y$  to mean  $x > y$ . This relation is neither reflexive nor symmetric, but it is transitive.

**Example 1.23** On  $\mathbb{Z}$ , define  $x \mathfrak{R} y$  to mean  $x \leq y$ . This is reflexive and transitive, but not symmetric.

**Example 1.24** On  $\mathbb{Z}$ , define  $x \mathfrak{R} y$  to mean  $x \leq y + 1$ . This relation is reflexive, but neither symmetric nor transitive.

**Example 1.25** On  $\mathbb{Z}$ , define  $x \mathfrak{R} y$  to mean  $x = -y$ . This relation is neither reflexive nor transitive, but it is symmetric.

**Example 1.26** On the set  $F$  of all fractions of the form  $a/b$  with  $a, b \neq 0$ , define  $a/b \mathfrak{R} c/d$  iff  $b = c$ . This relation is neither reflexive nor symmetric nor transitive.

**Example 1.27** Let  $A$  be the set  $\{1, 2, 4, 6, \dots\}$ . Define the relation  $\mathfrak{R}$  by saying  $x \mathfrak{R} y$  iff  $x$  and  $y$  have a common factor other than 1. This relation is symmetric and transitive, but it is not reflexive, because  $1 \mathfrak{R} 1$  is not true.

**Example 1.28** On  $\mathbb{C}$ , the set of complex numbers, define  $z \mathfrak{R} w$  to mean  $\operatorname{Re}(z) \leq \operatorname{Re}(w)$  and  $\operatorname{Im}(z) \leq \operatorname{Im}(w)$ . It is reflexive and transitive, but not symmetric.

### Problem Set 1.3

1. Determine whether each of the relations defined below on the set  $S$  is reflexive, symmetric or/and transitive. Pick out the equivalence relations.

- (a)  $x \mathfrak{R} y \Leftrightarrow x \leq y + 1 : S = \mathbb{R}$
- (b)  $x \mathfrak{R} y \Leftrightarrow x = 2y : S = \mathbb{Q}$
- (c)  $x \mathfrak{R} y \Leftrightarrow x - y$  is divisible by 2 :  $S = \mathbb{Z}$
- (d)  $x \mathfrak{R} y \Leftrightarrow x$  is the brother of  $y$  :  $S$  is the set of all human beings
- (e)  $x \mathfrak{R} y \Leftrightarrow x \subset y : S$  is the set of all subsets of  $\mathbb{R}$
- (f)  $x \mathfrak{R} y \Leftrightarrow x$  is married to  $y$  :  $S$  is the set of all human beings
- (g)  $x \mathfrak{R} y \Leftrightarrow x = |y| : S = \mathbb{R}$
- (h)  $x \mathfrak{R} y \Leftrightarrow x = -y : S = \mathbb{Q}$
- (i)  $x \mathfrak{R} y \Leftrightarrow \sin x = \sin y : S = \mathbb{R}$
- (j)  $z \mathfrak{R} w \Leftrightarrow |z| = |w| : S = \mathbb{C}$
- (k)  $x \mathfrak{R} y \Leftrightarrow x$  and  $y$  are students of BITS :  $S$  is the set of all students of India.

## 1.4 FUNCTIONS

The reader must have already been exposed, in his earlier training, to a proper definition of functions and operations with functions. However, as the concept of a function is very fundamental to mathematics and its applications, we shall now give the basic ideas about functions relevant to our subject.

**1.4.1 Definition** Let  $A$  and  $B$  be two nonempty sets. A *function (map)*  $f$  from  $A$  to  $B$  is a rule which, to each element of  $A$ , associates a unique element of  $B$ . Symbolically, we write

$$f: A \rightarrow B.$$

For each  $x \in A$ , the unique element of  $B$  associated with  $x$  is denoted by  $f(x)$  and is called the *image of  $x$  by (under)  $f$*  or the  *$f$ -image of  $x$*  or the *value of  $f$  at  $x$* . We also say that  $x$  is mapped into  $f(x)$  by  $f$ .  $x$  itself is said to be a *pre-image of  $f(x)$  under  $f$* .  $A$  is called the *domain of  $f$*  written as  $D(f)$ . It is the set on which  $f$  is defined.  $B$  is called the *target set of  $f$* . It is the set to which the images belong.

For clarity, we often draw a diagram as in Figure 1.2.

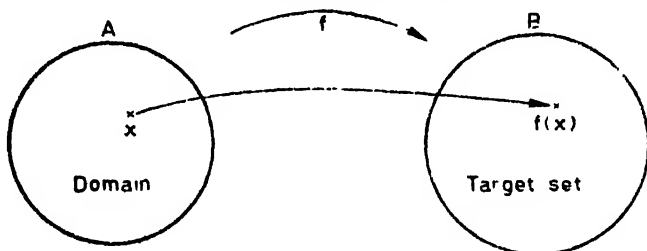


FIGURE 1.2

We shall set up a number of conventions on usage, by means of a simple example of a function: Consider the function  $f: R \rightarrow R$  defined by the rule, which, to each  $x \in R$ , associates the value  $x^2$ . So  $f(x) = x^2$  for all  $x \in R$ . By abuse of language, we sometimes say that the function is  $x^2$ , and we even use the symbol  $f(x)$  for the function. Very often, we use the more precise language that the function is given by  $f(x) = x^2$ ,  $x \in R$ . Alternatively, we denote this function as the set  $\{(x, x^2) \mid x \in R\}$  of ordered pairs. This set is also called the *graph of the function*; because, if we plot the points  $(x, x^2)$  in a plane, we will get the geometrical curve representing the function.

We shall now illustrate the foregoing ideas by listing seven ways of expressing this  $x^2$ -function. The first four methods are technically perfect, and methods (v) to (vii) are sanctified by custom and usage.

- (i)  $f: R \rightarrow R$  defined by  $x \mapsto x^2$ .
- (ii)  $f: x \mapsto x^2$  for all  $x \in R$ .

- (iii)  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$ .
- (iv)  $f = \{(x, x^2) \mid x \in \mathbb{R}\}$ .
- (v) The function  $x^2$  defined on  $\mathbb{R}$ .
- (vi) The graph of the function is  $\{(x, x^2) \mid x \in \mathbb{R}\}$ .
- (vii) The graph of the function is as in Figure 1.3.

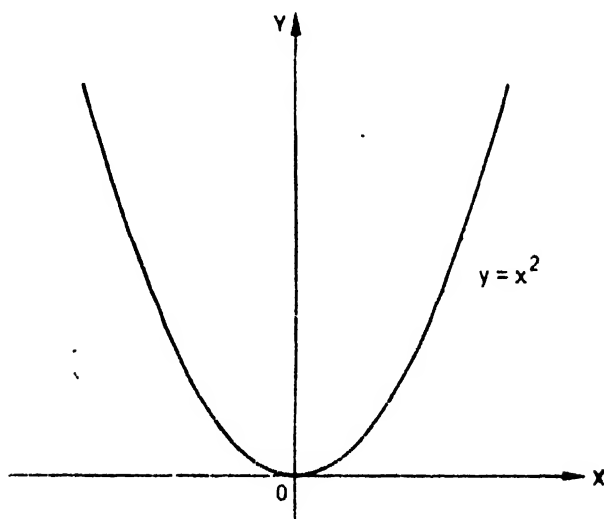


FIGURE 1.3

Note that the symbol ' $\mapsto$ ' used in (i) and (ii) is different from the symbol ' $\rightarrow$ ' for a function. The symbol ' $\rightarrow$ ' is used when we want to specify the domain and target set of  $f$ , whereas ' $\mapsto$ ' is used when we want to indicate the image of a single element of the domain. Thus,  $x \mapsto x^2$  means that the image of  $x$  is  $x^2$ . On the other hand, when we write  $f: A \rightarrow B$ , we are not specifying the image of a single element.

Though we shall allow ourselves the liberty of using any of these methods to specify a function, in the sequel, more often than not, we shall adhere to methods (i), (ii), and (iii). In case there is doubt while using the other methods, the reader should go back to the first three methods for clarification.

It may be noted that, if it is possible to list, without ambiguity, all the values of  $f(x)$  as  $x$  varies in the domain, we do so, and we also say that this list is the function.

Now we emphasise the word 'unique' occurring in the definition of a function. For each  $x \in A$ ,  $f(x)$  should be unique. In other words,  $f$  cannot have two or more values at a given  $x$ . For example, if  $A = \{1, 2, 3, 4, 5\}$  and  $B = \{-1, 2, -3, 4, -5, 6, 0\}$ , then the association  $1 \mapsto -1$ ;

$2 \mapsto 2$  and  $0$ ;  $3 \mapsto -3$ ;  $4 \mapsto 0$ ,  $5 \mapsto -5$  is not a function, since the image of 2 is not unique.

**Example 1.29**  $A = \{1, 2, 3, 4, 5, 6\}$  and  $B = \{-1, 2, -3, 4, -5, 6, 0\}$ . Define  $f: A \rightarrow B$  by saying  $f(1) = -1$ ,  $f(2) = 2$ ,  $f(3) = -3$ ,  $f(4) = 4$ ,  $f(5) = -5$ , and  $f(6) = 6$ . This function is nothing but  $\{(1, -1), (2, 2), (3, -3), (4, 4), (5, -5), (6, 6)\}$ .

**Example 1.30** Given  $A$  and  $B$  as in Example 1.29, define  $f: A \rightarrow B$  as follows:  $f(1) = -1$ ,  $f(2) = 0$ ,  $f(3) = -3$ ,  $f(4) = 0$ ,  $f(5) = -5$ ,  $f(6) = 0$ . This function is nothing but  $\{(1, -1), (2, 0), (3, -3), (4, 0), (5, -5), (6, 0)\}$ .

**Example 1.31** Define  $f: \mathbb{R} \rightarrow \mathbb{R}$  by  $f(x) = |x|$ . This is called the *absolute value function*.

**Example 1.32** Define  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  by  $f(x_1, x_2) = x_1 + x_2$ . To each point  $(x_1, x_2)$  in  $\mathbb{R}^2$ , this function associates the real number  $x_1 + x_2$ . In other words, to every pair  $(x, y)$  of real numbers, this function associates their sum  $x + y$ . Therefore, this function is called *addition* in  $\mathbb{R}$ .

**Example 1.33** In the same manner as in Example 1.32, *multiplication* in  $\mathbb{R}$  can be written as  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $f(x_1, x_2) = x_1 x_2$ .

**Example 1.34** Fix a real number  $\lambda$ . Then  $f: x \mapsto \lambda x$  for all  $x \in \mathbb{R}$  is a function, called (*scalar*) *multiplication* by  $\lambda$ .

**Example 1.35** Let  $\alpha$  be a fixed real number. Then  $f: x \mapsto x + \alpha$  for all  $x \in \mathbb{R}$  is a function, called *translation* by  $\alpha$ .

**Example 1.36**  $f: x \mapsto 0$  for all  $x \in \mathbb{R}$  is a function, called the *zero function* or the zero polynomial on  $\mathbb{R}$ . It is denoted by 0. Note that  $f(x) = 0$  for all  $x \in \mathbb{R}$  and its graph is just the  $x$ -axis.

**Example 1.37** Let  $A$  and  $B$  be two nonempty sets and  $b_0 \in B$  be a fixed element. Then the function  $f: A \rightarrow B$  defined by  $f(x) = b_0$  for all  $x \in A$  is called a *constant function*. Note that Example 1.36 is a special case of this function.

**Example 1.38** The function  $f: A \rightarrow A$  defined by  $f(x) = x$  for all  $x \in A$  is called the *identity function* on  $A$  and is usually denoted by  $I_A$ .

**Example 1.39** Let  $n$  be a nonnegative integer and  $a_0, a_1, \dots, a_n$  be fixed elements in  $\mathbb{R}$ . Then the function

$$p: x \mapsto a_0 + a_1 x + \dots + a_n x^n, x \in \mathbb{R},$$

is called a *real polynomial function* on  $\mathbb{R}$  or simply a *real polynomial* on  $\mathbb{R}$ . Note that it is a map from  $\mathbb{R}$  to  $\mathbb{R}$ . By abuse of language,  $a_0 + a_1 x + \dots + a_n x^n$  itself is called a polynomial. To denote the fact that  $a_0, a_1, \dots, a_n$  are real, it is also called a *polynomial with real coefficients*.

If  $a_n \neq 0$ , then  $n$  is said to be the *degree* of the polynomial  $a_0 + a_1 x + \dots + a_n x^n$ . The set of all real polynomials is denoted by  $\mathcal{P}$ , and the

set of all real polynomials of degree not greater than  $n$  is denoted by  $\mathcal{P}_n$ . Note that the zero polynomial, 0, belongs to both  $\mathcal{P}$  and  $\mathcal{P}_n$ . The degree of the zero polynomial is, by convention, assigned to be  $-\infty$ . Similarly,  $p: C \rightarrow C$  given by  $p(z) = a_0 + a_1z + \dots + a_nz^n$ , where  $a_0, a_1, \dots, a_n$  are complex numbers, is called a *complex polynomial* on  $C$ .

**Example 1.40** Define  $f: \mathcal{P} \rightarrow R$  by  $f(p) = p(0)$  for all  $p \in \mathcal{P}$ . For instance, if the polynomial is  $p: x \mapsto 2x^2 + 1$ , then  $f(p) = 1$ . Note that this is not a constant function. (Why?)

**Example 1.41** Let  $N$  be the set of natural numbers. Then any function  $f: N \rightarrow R$  can be written as the set of ordered pairs  $\{(1, f(1)), (2, f(2)), (3, f(3)), \dots\}$ . Such a function is called a *sequence* of real numbers. The sequence is also written as

$$f(1), f(2), f(3), \dots$$

For instance,  $2, 4, 6, 8, \dots$  is a sequence. It is just the function  $x \mapsto 2x$  on  $N$ .

In the discussion of any function  $f$ , there are four fundamental questions:

- (i) What is the domain of  $f$ ?
- (ii) What is the range of  $f$ ?
- (iii) Is the function 'onto' or not?
- (iv) Is the function 'one-one' or not?

We have already introduced the domain of  $f$ . We shall now introduce the concepts related to other questions.

**1.4.2 Definition** Let  $f: A \rightarrow B$  be a function and  $y \in B$ . The *pre-image* of  $y$  is the set  $\{x \in A \mid f(x) = y\}$ .

**1.4.3 Remark** An accepted notation for the pre-image of  $y$  under  $f$  is  $f^{-1}(y)$ . However, we shall not use this notation in this sense.

**Example 1.42** Define  $f: R \rightarrow R$  by the rule  $f(x) = x^2$ . The pre-image of 4 is  $\{2, -2\}$ , and the pre-image of 0 is  $\{0\}$ . The pre-image of  $-1$  is the empty set  $\phi$ .

**1.4.4 Definition** Let  $f: A \rightarrow B$  be a function. The set  $\{f(x) : x \in A\}$  is called the *range* of  $f$  and is denoted by  $R(f)$ .

It is in fact the set of all  $f$ -images.

**1.4.5 Definition** A map  $f: A \rightarrow B$  is said to be *onto* (surjective) if  $R(f) = B$ .

Clearly, the following statement is true.

**1.4.6 Fact** A map  $f: A \rightarrow B$  is onto iff either one of the following properties holds:

- (a) For every  $y \in B$ , there exists at least one  $x \in A$  such that  $f(x) = y$ .

(1)



- (b) For every  $y \in B$ , the pre-image of  $y$  is a nonempty subset of  $A$ . (2)

The proof of this fact is left to the reader.

We now consider Examples 1.29 to 1.41 with reference to the foregoing definitions. The function  $f: x \rightarrow x^2$  from  $R$  to  $R$  is not onto, because the negative real numbers have empty pre-images.

In Example 1.29  $R(f) = B \setminus \{0\}$ . So  $f$  is not onto as also the function in Example 1.30, because

$$R(f) = \{-1, -3, -5, 0\}.$$

In Example 1.31  $R(f) = \{x \mid x \geq 0\}$ . So  $f$  is not onto. But in Example 1.32 the range is  $R$ , since every number in  $R$  can be expressed as  $x_1 + x_2$  for suitable real numbers  $x_1$  and  $x_2$  in at least one way. Hence, the map is onto.

The function of Example 1.31 is onto because the range is  $R$ . The reason is similar to the one given for Example 1.32.

In Example 1.34, if  $\lambda \neq 0$ , the range is  $R$ , because every number in  $R$  can be expressed as  $\lambda x$  for a suitable  $x$  in  $R$ . The map is onto in this case. If  $\lambda = 0$ , the range is  $\{0\}$  and the map is not onto.

In Example 1.35 the map is onto, since the range is  $R$ . (The reader should reason this out.) But the map of Example 1.36 is not onto, as the range is  $\{0\}$ . This is an extreme case of a function not being onto.

The map in Example 1.37 is, in general, not onto as the range is  $\{b_0\}$ . It becomes an onto map if  $B$  is a singleton. In Example 1.38, since the range is  $A$ , the map is onto.

In Example 1.39, unless  $p$  is specified, the range cannot be calculated. For example, the polynomial  $p: x \mapsto x + 1$  is an onto function, whereas the polynomial  $q: x \mapsto x^2 + 1$  is not. (Why?)

In Example 1.40 the range is  $R$  (why?), so the map is onto, whereas in Example 1.41 the map is not onto, because the range is  $\{f(1), f(2), \dots\}$ . It is a proper subset of  $R$ . The proof of this is beyond the scope of our treatment.

**1.4.7 Definition** A map  $f: A \rightarrow B$  is said to be *one-one* (injective) if distinct elements of  $A$  are mapped into distinct elements of  $B$ .

The function  $x \mapsto x^2$ ,  $x \in R$  is not one-one, because we can find several pairs of distinct elements  $x_1, x_2$  that have the same image, e.g. both 2 and -2 map into 4.

Clearly, the following statement is true. The proof is left to the reader.

**1.4.8 Fact** A map  $f: A \rightarrow B$  is one-one iff any one of three properties holds:

$$(a) \quad x \neq y \Rightarrow f(x) \neq f(y). \quad (3)$$

$$(b) \quad f(x) = f(y) \Rightarrow x = y. \quad (4)$$

- (c) The pre-image of each element in the range of  $f$  is a singleton. (5)

Note that whenever we want to prove that a certain function is one-one, we have to prove (3) or (4) or (5).

**1.4.9 Remark** The name 'one-one' comes as follows : Already in the definition of a function, given  $x$ ,  $f(x)$  is unique. The requirement of Definition 1.4.7 further asserts that, given  $f(x)$ ,  $x$  is unique. Thus, the correspondence  $x \mapsto f(x)$  is unique both ways. Such a correspondence is called a one-one correspondence. Hence the name one-one function.

Let us now check which of the examples from 1.29 to 1.41 are one-one.

In Example 1.29 the function is one-one. In Example 1.30 it is not, because the pre-image of 0 is  $\{2, 4, 6\}$ .

In Example 1.31  $f$  is not one-one, because for any  $x \in R$ ,  $x$  and  $-x$  both map into the same image.

In Example 1.32  $f$  is not one-one, because a number can be expressed as the sum of two real numbers in more than one way. For instance,  $7 = 3 + 4 = 5 + 2$ . So  $f(3, 4) = 7 = f(5, 2)$ . For a similar reason, the multiplication function of Example 1.33 is not one-one.

In Example 1.34, if  $\lambda \neq 0$ , (scalar) multiplication by  $\lambda$  is one-one, because  $\lambda x_1 = \lambda x_2 \Rightarrow x_1 = x_2$ , which proves (4) in Fact 1.4.8. If  $\lambda = 0$ , the map is clearly not one-one.

In Example 1.35 translation by ' $a$ ' is a one-one map, because  $x_1 + a = x_2 + a \Rightarrow x_1 = x_2$ .

The maps of Examples 1.36 and 1.37 are clearly not one-one, because all members of  $A$  are mapped into one member of  $B$ . In fact, this is the extreme case of not being one-one. On the other hand, the map of Example 1.38 is one-one.

In Example 1.39 whether or not the function is one-one depends upon the polynomial, because the polynomial  $p : x \mapsto x + 1$  is one-one, whereas the polynomial  $q : x \mapsto x^2 + 1$  is not one-one. (Why ?)

The map of Example 1.40 is not one-one, because  $(x+1)$  and  $(x^2+1)$  have the same image.

In Example 1.41 also the one-oneness depends upon the sequence, e.g. the sequence  $1, 1/2, 1/3, 1/4, \dots$  is one-one, whereas the sequence  $1, 1, 1, \dots$  is not one-one.

**1.4.10 Definition** Two sets  $A$  and  $B$  are said to be in *one-one correspondence* if there exists a map  $f : A \rightarrow B$  that is one-one and onto.

**1.4.11 Definition** If a set  $A$  is in one-one correspondence with the set  $\{1, 2, \dots, n\}$ , for some positive integer  $n$ ,  $A$  is said to be a *finite set*.

Sets that are not finite are called *infinite sets*.

For example, the set of all real roots of the equation  $x^3 - 2x^2 - x + 2 = 0$  and the set of vertices of the triangle  $ABC$  are in one-one correspondence. Both are finite sets, since they are in one-one correspondence with the set  $\{1, 2, 3\}$ .

$N$ ,  $Q$ ,  $R$ ,  $Z$ , and  $C$  are all infinite sets. (Why ?)

### Problem Set 1.4

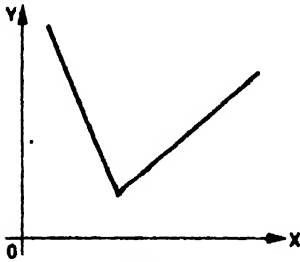
- Determine which of the following subsets of  $R \times R$  are functions :
  - $\{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5)\}$
  - $\{(1, 1), (2, 1), (3, 1), (4, 1), (5, 1)\}$
  - $\{(1, \pi), (\pi, 1), (0, e), (e, 1)\}$
  - $\{(\pi, e), (\pi, 1), (1, e), (e, 1)\}$ .
- For the given sets  $A$  and  $B$ , define, if possible, a function  $f: A \rightarrow B$  such that (i)  $f$  is onto but not one-one, (ii)  $f$  is neither one-one nor onto, (iii)  $f$  is one-one but not onto, (iv)  $f$  is one-one and onto :
  - $A = \{a, b, c, d\}$ ,  $B = \{0, 1, 2, 3\}$
  - $A = \{1, 2, 3\}$ ,  $B = \{0, 1, 2, 3\}$
  - $A = \{0, 1, 2, 3\}$ ,  $B = \{1, 2, 3\}$
  - $A = R$ ,  $B = R$ .
- Determine the largest subset  $A$  of  $R$  for the function  $f: A \rightarrow R$  defined as follows :
 

(a) $f(x) = \sqrt{(x^2 - 1)}$	(b) $f(x) = \sqrt{(4 - x^2)}$
(c) $f(x) = x/(1 + x)$	(d) $f(x) = x^2 - 2x + 7$
(e) $f(x) = x/(x^2 + 1)$	(f) $f(x) = \sqrt{(x^3 - x^2)}$ .
- Determine the largest subset  $A$  of  $C$  for the function  $f: A \rightarrow C$  defined as follows :
 

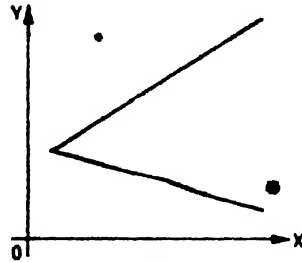
(a) $f(z) =  z $	(b) $f(z) = z/ z $
(c) $f(z) = \operatorname{Re}(z)$	(d) $f(z) = \exp(i \operatorname{Im}(z))$
(e) $f(z) = \exp(\operatorname{Re}(z))$	(f) $f(z) = \bar{z}$ .
- Determine the range of each of the functions in Problems 4 and 5.
- Determine which of the functions in Problem 4 are one-one and which onto.
- True or false ?
  - There is a one-one correspondence between  $A \times B$  and  $B \times A$ .
  - There is a one-one correspondence between  $C$  and  $R \times R$ .
  - If  $A$  and  $B$  are finite sets, then every one-one map  $f: A \rightarrow B$  is also onto.

- (d) If  $A$  and  $B$  are finite sets, then every onto map  $f: A \rightarrow B$  is also one-one.  
 (e) A nonempty subset of a finite set is also finite.  
 (f) Every nonempty subset of an infinite set is infinite.

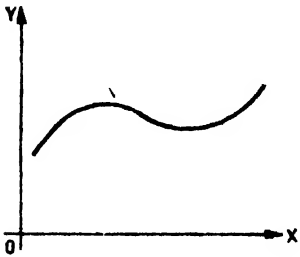
8. Determine which of the following graphs represent functions :



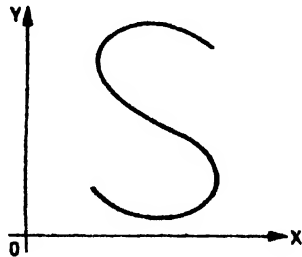
(a)



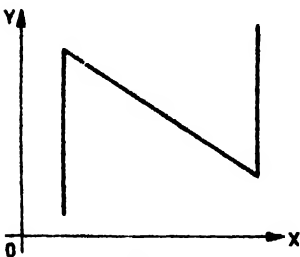
(b)



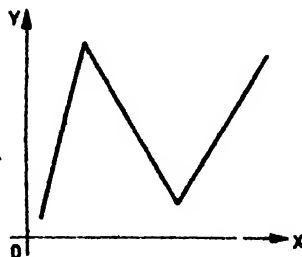
(c)



(d)



(e)



(f)

## 1.5 BINARY OPERATIONS

We shall now introduce a very important class of functions which includes the familiar operations of arithmetic, namely, addition and multiplication. If we look back carefully at the operation of addition of numbers, we find it is essentially a process of combining two numbers and obtaining another number in an unambiguous way. The same is true of multiplication. A generalisation of this idea is the following definition.

**1.5.1 Definition** Given a nonempty set  $A$ , any function from  $A \times A$  to  $A$  is called a *binary operation* on  $A$ .

In other words, to each ordered pair  $(x, y)$  of elements  $x$  and  $y$  of  $A$ , the binary operation  $*$  on  $A$  associates a unique element  $z$  of  $A$ . We write  $z = x * y$ .

The functions defined in Examples 1.32 and 1.33 are binary operations on  $R$ . But the (scalar) multiplication by a fixed  $\lambda$  in Example 1.34 is not a binary operation on  $R$ , because it is not a function on  $R \times R$ . It does not associate a number with every pair of numbers in  $R$ .

Note that though addition is a binary operation on  $R$  or  $N$  or  $Q$  or the set  $E$  of even integers, it is not a binary operation on the set of odd integers, because the sum of two odd integers is not an odd integer. We formalise this situation by the following definition.

**1.5.2 Definition** Let  $B \subset A$ . Let  $*$  be a binary operation on  $A$ . If for each pair of elements  $(x, y)$  in  $B$ ,  $x * y$  belongs to  $B$ , we say that  $B$  is closed under ' $*$ '. If there exist  $x, y$  in  $B$  such that  $x * y \notin B$ , we say that  $B$  is not closed under ' $*$ '.

Note that the situation ' $B$  is (not) closed under ' $*$ ' is sometimes referred to as ' $*$  is (not) a closed operation in  $B$ '.

To illustrate Definition 1.5.2, let us consider a couple of examples.

**Example 1.43** The set of odd integers is not closed under the usual addition of numbers, but it is closed under multiplication.

**Example 1.44** The set  $N$  is not closed under division, but it is closed under addition.

The reader should note that unless  $B$  is closed under ' $*$ ', it cannot be a binary operation on  $B$ . We now give some examples of binary operations.

**Example 1.45**  $f : (x, y) \mapsto x - y$  is a binary operation on  $R$ . It is also a binary operation on  $Q$ ,  $Z$ , and  $C$ , but it is not a binary operation on  $N$ .

**Example 1.46**  $f : (x, y) \mapsto x/y$  is a binary operation on  $R \setminus \{0\}$ . We cannot have the whole set  $R$  here, because  $x/y$  is not defined when  $y = 0$ . It is also a binary operation on  $C \setminus \{0\}$ ,  $Q \setminus \{0\}$ , but not on  $Z \setminus \{0\}$  or  $N$ . (Why?)

**Example 1.47**  $f : (x, y) \mapsto x + y - xy$  is a binary operation on  $C$ ,  $R$ ,  $Q$ , and  $Z$ , but not on  $N$ . (Why?)

**Example 1.48**  $f : (x, y) \mapsto x^y$  is a binary operation on  $N$ , but not on  $Z$  or  $Q$  or  $R$ . (Why?)

**Example 1.49**  $f : (x, y) \mapsto |x - y|$  is a binary operation on  $Q$ ,  $Z$ ,  $R$ , and  $C$ .

**1.5.3 Definition** A binary operation  $*$  on a set  $A$  is said to be

- (a) *commutative* if  $a * b = b * a$  for all  $a, b \in A$ ;

(b) *associative* if  $a * (b * c) = (a * b) * c$  for all  $a, b, c \in A$ .

Whenever ' $*$ ' is associative, we can write  $a * b * c$  in place of  $a * (b * c)$  or  $(a * b) * c$ .

The usual addition and multiplication of numbers are both commutative and associative. The operations defined in Examples 1.45, 1.46, and 1.48 are neither commutative nor associative. But in Example 1.47 the operation is both commutative and associative. In Example 1.49 the operation is commutative, but not associative. The reader should check these statements carefully.

We shall not discuss nonassociative binary operations in this book, but we shall deal with some important noncommutative binary operations.

### Problem Set 1.5

1. Define addition ' $+$ ' and multiplication ' $\times$ ' on the set  $S = \{\alpha, \beta, \gamma\}$  with the help of Tables 1.1 and 1.2.

TABLE 1.1

$+$	$\alpha$	$\beta$	$\gamma$
$\alpha$	$\alpha$	$\beta$	$\gamma$
$\beta$	$\beta$	$\gamma$	$\alpha$
$\gamma$	$\gamma$	$\alpha$	$\beta$

TABLE 1.2

$\times$	$\alpha$	$\beta$	$\gamma$
$\alpha$	$\alpha$	$\alpha$	$\alpha$
$\beta$	$\alpha$	$\beta$	$\gamma$
$\gamma$	$\alpha$	$\gamma$	$\beta$

Prove that the binary operations ' $+$ ' and ' $\times$ ' as defined by these tables are commutative and associative.

2. Define the binary operation ' $*$ ' on  $R$ , the set of real numbers, as follows :

$$x * y = xy + x + y.$$

- Is ' $*$ ' commutative? Is ' $*$ ' associative?
3. Give an example of a binary operation, which is
  - (a) associative but not commutative
  - (b) commutative but not associative
  - (c) neither commutative nor associative.
4. Define addition ' $+$ ' and multiplication ' $\times$ ' in  $R \times R$  as follows :

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$(x_1, x_2) \times (y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Prove that these are commutative and associative binary operations,

given that  $(x_1, x_2) = (y_1, y_2)$  means  $x_1 = y_1$  and  $x_2 = y_2$ .

5. Define the binary operations '+' and '×' on  $R \times R$  as follows :

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$(x_1, x_2) \times (y_1, y_2) = (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1).$$

Prove that both these operations are commutative as well as associative. (Equality is defined as in Problem 4.)

6. Let  $Z_0$  be the set of nonzero real numbers. Define addition '+' and multiplication '×' on  $Z \times Z$  as follows :

$$(x_1, x_2) + (y_1, y_2) = (x_1 y_2 + x_2 y_1, x_2 y_2)$$

$$(x_1, x_2) \times (y_1, y_2) = (x_1 y_1, x_2 y_2).$$

Prove that these are commutative and associative binary operations, given that  $(x_1, x_2) = (y_1, y_2)$  means  $x_1 y_2 = x_2 y_1$ .

## 1.6 ALGEBRAIC STRUCTURES

### FIELDS

The standard properties of real numbers with respect to addition and multiplication are so fundamental in mathematics that whenever any set  $X$  with two binary operations '+' and '·' (called addition and multiplication) satisfies these properties, we give a special name to such a system. It is called a *field*. To understand these properties, which are listed in the following formal definition, the reader is advised to think of  $X$  as the set  $R$  of real numbers and the operations '+' and '·' as the usual addition and multiplication in  $R$ .

- 1.6.1 Definition** Let  $X$  be a nonempty set on which there are two binary operations '+' and '·', called addition and multiplication, respectively. Then the set  $X$  together with these operations is said to be a *field*, if the following axioms are satisfied :

**F1** Addition is associative.

**F2** There exists an element  $0 \in X$  with the property that  $0 + a = a + 0 = a$  for all  $a \in X$ .

**F3** For each  $a \in X$ , there exists an  $x \in X$  such that  $a + x = 0 = x + a$ .

**F4** Addition is commutative.

**F5** Multiplication is associative.

**F6** There exists an element  $1 \in X$  such that  $1 \cdot a = a = a \cdot 1$  for all  $a \in X$ .

**F7** For each  $a (\neq 0)$  in  $X$ , there exists an element  $y \in X$  such that  $a \cdot y = 1 = y \cdot a$ .

**F8** Multiplication is commutative.

**F9** Multiplication is distributive over addition, that is, for all  $a, b, c \in X$ ,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

and

$$(a + b) \cdot c = a \cdot c + b \cdot c.$$

In the case of  $X = R$  the element  $x$  mentioned in F3 is just the familiar  $-a$ , the negative of  $a$ , and the element  $y$  mentioned in F7 is just the familiar  $1/a$ , the reciprocal of  $a$ .

Axioms F1 to F9, which are valid in  $R$ , are fundamental. Starting from them, all the formulae of elementary algebra can be derived.

The classical examples of fields are, besides  $R$ ,

(i) the set  $Q$  of rational numbers under the usual addition and multiplication and

(ii) the set  $C$  of complex numbers under the usual addition and multiplication.

In this book we shall be interested in only these three fields, namely,  $R$ ,  $Q$ , and  $C$ .

$Z$  is not a field under addition and multiplication, because F7 is not satisfied. So also  $N$  is not a field. (Why?)

## GROUPS

Now we take up another structure in algebra, which is simpler than a field. Suppose we restrict our attention to only one binary operation on a nonempty set  $X$  and denote it by '+'. If this satisfies axioms F1, F2, and F3, we say that the system is a group. If, in addition, F4 is satisfied, we call it a commutative group. We shall now give the formal definition.

**1.6.2 Definition** Let  $G$  be a nonempty set on which a binary operation '\*' is defined. Then  $G$  is said to be a *group*, under the operation '\*', if the following axioms are satisfied :

**G1** '\*' is associative.

**G2** There exists an element  $e \in G$  such that  $e * a = a = a * e$  for all  $a \in G$ .  $e$  is called an *identity* for '\*'. .

**G3** For each  $a \in G$ , there exists an element  $x \in G$  such that  $a * x = e = x * a$ .  $x$  is said to be an *inverse* of  $a$  for the operation '\*'. .

If, in addition,

**G4** The operation '\*' is commutative,  
then the group is said to be *commutative* or *abelian*.



Let us consider four examples :

- (i)  $R$ ,  $Q$ , and  $Z$  are groups under the usual addition operation.
- (ii)  $N$ , under addition, is not a group, because G2 and G3 are not satisfied.
- (iii)  $Q \setminus \{0\}$  and  $R \setminus \{0\}$  are both groups under the operation of usual multiplication. The number 1 plays the role of 'e' in the definition.
- (iv) The set  $Z \setminus \{0\}$ , under multiplication, is not a group, because G3 is not satisfied.

In all these four examples of groups the operation is commutative. So they are all commutative groups. However, noncommutative groups are of great importance in mathematics and its applications. Watch for some examples of noncommutative groups as we go along.

Before we proceed, we shall establish the uniqueness of the 'identity' referred to in G2 and the 'inverse' referred to in G3.

**1.6.3 Theorem** (a) *In a group  $G$  there is only one identity, i.e. the identity is unique.*

(b) *For a given element  $a$  in a group  $G$  there is only one inverse, i.e. the inverse of an element  $a \in G$  is unique.*

*Proof:* (a) Let  $e$  and  $e'$  be two elements in  $G$ , both having the property stated in G2, namely,

$$e * a = a = a * e \quad \text{for all } a \in G \quad (1)$$

and

$$e' * a = a = a * e' \quad \text{for all } a \in G. \quad (2)$$

Then taking  $e'$  for  $a$  in Equation (1), we get

$$e * e' = e' * e = e',$$

and taking  $e$  for  $a$  in Equation (2), we get

$$e' * e = e * e' = e.$$

Hence,  $e = e'$ , i.e. the identity is unique.

(b) Given  $a \in G$ , let  $x$  and  $y$  be two elements in  $G$ , both having the property stated in G3, namely,

$$a * x = x * a = e \quad (3)$$

and

$$a * y = y * a = e. \quad (4)$$

Therefore,

$$\begin{aligned} y &= y * e && \text{(by G2)} \\ &= y * (a * x) && \text{(by (3)).} \end{aligned}$$

But the operation '\*' is associative in  $G$ . Hence, we get

$$\begin{aligned} y &= (y * a) * x \\ &= e * x && \text{(by (4))} \\ &= x && \text{(by G2).} \end{aligned}$$

This proves that the inverse of each element is unique. ■

Thus, hereafter, we shall speak of *the identity* in a group  $G$  and also of *the inverse* of an element  $a \in G$ .

Whenever the operation is denoted by '+', the identity is denoted by '0', and the inverse of 'a' is denoted by '-a'. It is customary to write  $a - b$  for  $a + (-b)$ .

## RINGS

Another important structure is that of a ring. Let us once again consider the nine field axioms. The first four axioms—F1, F2, F3, and F4—are pertinent to addition, whereas the next four axioms—F5, F6, F7, and F8—are pertinent to multiplication. The ninth axiom, F9, pertains to both addition and multiplication.

If there is one binary operation on a set  $X$  and the four axioms F1 to F4 are satisfied, then the system is a commutative group. If a set  $X$  has two binary operations and all the nine axioms are satisfied, we say the system is a field. In between these two situations, we have a ring structure.

Suppose a set  $X$  has two binary operations. Let, for the first operation, the system be a commutative group. The minimum that we require of the second operation, in order to get a meaningful system with two operations, is the associativity F5 and the distributivity F9.

So whenever there are two operations, say '+' and '·', on a set  $X$ , and F1 to F4, and F5 and F9 are satisfied, we call such a system a *ring*.

We shall not give a formal treatment of rings in this book. It suffices to note that

(i) A set  $X$  with two operations '+' and '·' such that (a) it is a commutative group for '+' and (b) F5 and F9 are satisfied is called a *ring*.

(ii) A ring in which F6 holds is called a *ring with identity* (unity).

(iii) A ring in which F8 holds is called a *commutative ring*.

For example,  $C$ ,  $R$ ,  $Q$ , and  $Z$  are rings under the usual operations of addition and multiplication. All these are commutative rings and have unity. The set  $E$  of even integers is a commutative ring under the operations of addition and multiplication. But it has no unity.

Watch for further examples of rings as we go along.

### Problem Set 1.6

1. In each of the Problems 1, 5, and 6 (Problem Set 1.5) prove that

the set under consideration, along with the operations of addition and multiplication defined therein, is a field.

2. Let  $G$  be an abelian group with addition '+' as the binary operation. Prove the following identities in  $G$ :
  - (a)  $-(-a - b) = a + b$       (b)  $a - 0 = a$
  - (c)  $-(a - b) = b - a$       (d)  $(c - b) - (a - b) = c - a$ .
3. Let  $A$  be a ring with '+' and  $\cdot$  as binary operations. Prove that, for  $a, b, c, d \in A$ .
  - (a)  $(-a) \cdot (-b) = a \cdot b$
  - (b)  $(a + b) \cdot (c + d) = a \cdot c + b \cdot d + a \cdot d + b \cdot c$ .
4. Prove that a ring contains at most one unity.
5. Prove that  $a \cdot 0 = 0$  for all  $a$  in a ring  $A$ .
6. Give three reasons why  $N$  is not a field under the usual addition and multiplication.

## 1.7 OPERATIONS ON FUNCTIONS

The standard operations on functions, which yield new functions from the given ones, are (i) composition, (ii) addition, (iii) scalar multiplication, and (iv) multiplication.

Before we begin a study of these operations, we have to make precise what we mean when we say 'two functions are equal or are the same'.

**1.7.1 Definition** Two functions  $f: A \rightarrow B$  and  $g: A \rightarrow B$  are said to be *equal*, written  $f = g$ , if  $f(x) = g(x)$  for all  $x \in A$ .

For example, the function  $x \mapsto x^2$  defined on  $[-1, 1]$  and the function  $x \mapsto x^2$  defined on  $[0, 1]$  are not equal, because their domains are different.

**1.7.2 Definition** Let  $f: A \rightarrow B$  and  $g: B \rightarrow C$  be two functions. Define the function  $h: A \rightarrow C$  by

$$h(x) = g(f(x)) \quad \text{for all } x \in A.$$

Then  $h$  is called the *composition* of  $g$  and  $f$  and is denoted by  $g \circ f$ , which is read as ' $g$  composite  $f$ '. Thus, we have  $(g \circ f)(x) = g(f(x))$  for all  $x \in A$ . Symbolically, the composition can be represented as

$$\begin{array}{ccccc} & f & & g & \\ A & \rightarrow & B & \rightarrow & C \\ & & & & g \circ f \\ & & & & A \rightarrow C \end{array}$$

**Example 1.50** Let  $f: R \rightarrow R$  be defined by  $x \mapsto x + 1$  and  $g: R \rightarrow R$  be defined by  $x \mapsto x^2$ . Then  $g \circ f$  is defined, and  $(g \circ f)(x) = g(f(x)) = g(x + 1) = (x + 1)^2$ . Hence,  $g \circ f: x \mapsto (x + 1)^2$  is a map from  $R$  to  $R$ .

**Example 1.51** Let  $f: R \rightarrow R$  be defined by  $x \mapsto x + 1$  and  $g: R^+ \rightarrow R$  be defined by  $x \mapsto \sqrt{x}$ . Here  $R^+$  is the set of all positive real numbers.

In this case  $g \circ f$  is not defined, but  $f \circ g$  is defined, since

$$R^+ \xrightarrow{g} R \xrightarrow{f} R = R^+ \xrightarrow{f \circ g} R.$$

and

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) \\ &= f(\sqrt{x}) = \sqrt{x} + 1. \end{aligned}$$

Thus,  $f \circ g: x \mapsto \sqrt{x} + 1$  is a map from  $R^+$  to  $R$ .

**Example 1.52** In some cases both  $f \circ g$  and  $g \circ f$  may be defined. In Example 1.50  $f \circ g$  is also defined and  $(f \circ g)(x) = f(g(x)) = f(x^2) = x^2 + 1$ .

Thus, in this case  $f \circ g: R \rightarrow R$  is defined by  $x \mapsto x^2 + 1$ , but  $g \circ f: R \rightarrow R$  is defined by  $x \mapsto (x + 1)^2$ .

These examples show that when  $f \circ g$  is defined,  $g \circ f$  may or may not be defined. Even when both  $f \circ g$  and  $g \circ f$  are defined, they need not be equal. In other words, *composition of functions is not commutative*.

**Notations** The set of all functions  $f$  from  $A$  to  $B$  is denoted by  $\mathcal{F}(A, B)$ . If  $B = R$ , the set of real numbers, then the function  $f: A \rightarrow R$  is said to be a *real-valued function*. The set of all real-valued functions on  $A$  is denoted by  $\mathcal{F}(A, R)$ . The notation  $\mathcal{F}_R(A)$  is also used to denote this set of real-valued functions. If  $R$  is understood from the context, the symbol  $\mathcal{F}(A)$  can be used. If  $A$  is an interval,  $\mathcal{F}(A)$  is written also as  $\mathcal{F}[a, b]$  or  $\mathcal{F}(a, b)$ .

**1.7.3 Definition** Let  $f: A \rightarrow R$  and  $g: A \rightarrow R$  be two real-valued functions. Then the sum  $f + g$  is defined as a real-valued function as follows:  $f + g: A \rightarrow R$  such that  $x \mapsto f(x) + g(x)$ . In other words,  $(f + g)(x) = f(x) + g(x)$  for all  $x \in A$ .

$f + g$ , as defined, is called the *pointwise sum* of  $f$  and  $g$ . According to this definition,  $f + g$  is also a member of  $\mathcal{F}(A)$ .

**Example 1.53** Let  $f: R \rightarrow R$  be defined by  $f(x) = x^2$  and  $g: R \rightarrow R$

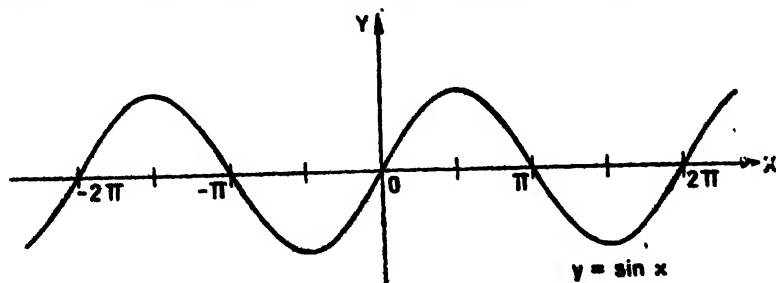


FIGURE 1.4

be defined by  $g(x) = x + 1$ . Then  $f + g$  is the function  $x \mapsto x^2 + x + 1$  for all  $x \in R$ .

**Example 1.54** Let us add the sine function and the cosine function. These functions are given by the graphs in Figures 1.4 and 1.5. Therefore, the

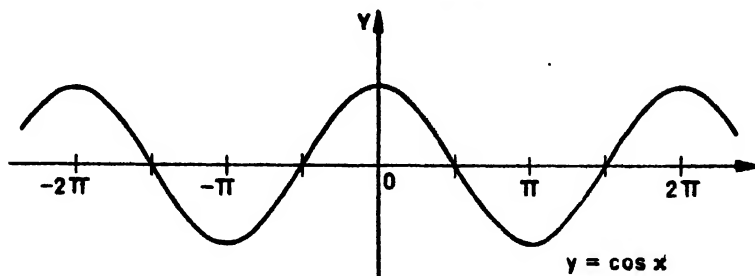


FIGURE 1.5

function sine + cosine is given by the rule (using Definition 1.7.3)

$$(\text{sine} + \text{cosine})(x) = \sin x + \cos x.$$

The graph of the function sine + cosine is as in Figure 1.6.

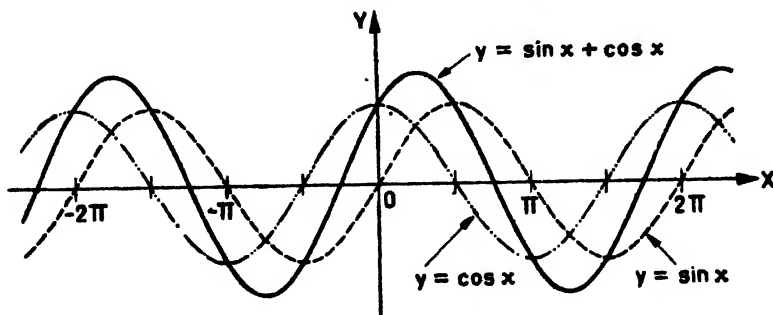


FIGURE 1.6

**1.7.4 Definition** Let  $\lambda$  be a real number and  $f: A \rightarrow R$  be a real-valued function. Then  $\lambda f$ , called the scalar multiple of  $f$  by  $\lambda$ , is defined as a real-valued function

$$\lambda f: A \rightarrow R \text{ such that } x \mapsto \lambda f(x).$$

In other words,  $(\lambda f)(x) = \lambda f(x)$  for all  $x \in A$ .  $\lambda f$  is called the *pointwise  $\lambda$ -multiple* of  $f$ . According to this definition,  $\lambda f$  is also a member of  $\mathcal{F}(A)$ .

**Example 1.55** Let  $f: R \rightarrow R$  be given by  $x \mapsto x^2$ . Then  $\lambda f$  is the function from  $R$  to  $R$  such that  $x \mapsto \lambda x^2$ .

**Example 1.56** The graph of the function  $1/2$  sine, i.e.  $\lambda$ -multiple of sine, with  $\lambda = 1/2$  is given in Figure 1.7.

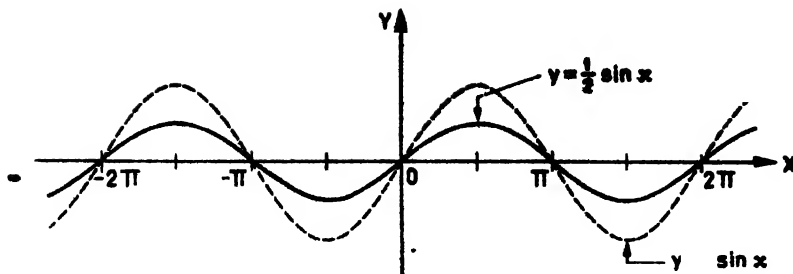


FIGURE 1.7

A special case of scalar multiplication when  $\lambda = -1$  gives  $(-1)f$  as the function

$$((-1)f)(x) = (-1)(f(x)) = -f(x) \quad \text{for all } x \in R.$$

$(-1)f$  is also denoted by  $-f$ .

**1.7.5 Definition** Let  $f: A \rightarrow R$  and  $g: A \rightarrow R$  be two real-valued functions. Then  $fg$ , called the *pointwise product* of  $f$  and  $g$ , is defined as a real-valued function

$$fg: A \rightarrow R \text{ such that } x \mapsto f(x)g(x).$$

In other words,  $(fg)(x) = f(x)g(x)$  for all  $x \in A$ .

**Example 1.57** Let  $f: R \rightarrow R$  be defined by  $x \mapsto x^2$  and  $g: R \rightarrow R$  be defined by  $x \mapsto x + 1$ . Then the function  $fg: R \rightarrow R$  is defined by  $x \mapsto x^2(x + 1)$ .

**Example 1.58** The graph of the product of sine and cosine is given in Figure 1.8.

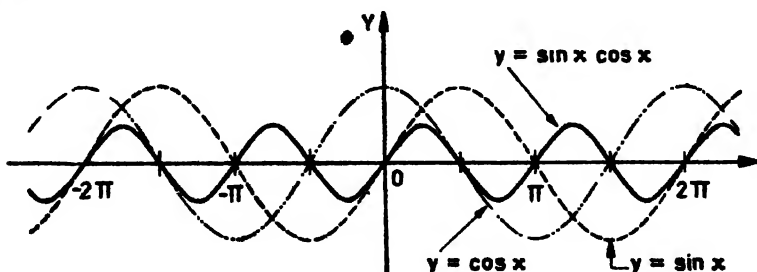


FIGURE 1.8

**1.7.6 Remark** We have constructed functions  $f + g$ ,  $\lambda f$ , and  $fg$ . The construction of  $f + g$  from  $f$  and  $g$  is nothing but an operation which, to each pair of functions  $f$  and  $g$  of  $\mathcal{F}(A)$ , associates a function  $f + g$  in  $\mathcal{F}(A)$ . Thus, this operation is a binary operation on  $\mathcal{F}(A)$ . It is called addition. The second operation, that of forming  $\lambda f$  from  $\lambda$  and  $f$ , called scalar multiplication, is not a binary opera-

tion on  $\mathcal{F}(A)$ . The reason is that we do not take two functions in  $\mathcal{F}(A)$  to produce a function in  $\mathcal{F}(A)$ . However, the multiplication of functions  $f$  and  $g$  is a binary operation on  $\mathcal{F}(A)$ .

We shall refer to Definitions 1.7.3, 1.7.4, and 1.7.5 as pointwise addition, pointwise scalar multiplication, and pointwise multiplication of functions, respectively. Throughout the book, without further explanation or mention, addition, multiplication by a scalar, or multiplication of functions is to be understood pointwise as already explained.

The following properties of addition in  $\mathcal{F}(A)$  are easy to check. They should not be taken for granted. The reader should carefully check them.

- (i) Addition in  $\mathcal{F}(A)$  is associative.
- (ii)  $G_2$  is satisfied for addition in  $\mathcal{F}(A)$ , because the zero function,  $0$ , acts as 'e' of  $G_2$ .
- (iii)  $G_3$  is satisfied for addition in  $\mathcal{F}(A)$ , because  $-f$  acts as the inverse of  $f$  for addition.
- (iv) Addition in  $\mathcal{F}(A)$  is commutative.

Thus,  $\mathcal{F}(A)$ , for pointwise addition, is a commutative group.

Analogously, we have, for multiplication, the following properties. The reader should carefully check them.

- (i) Multiplication in  $\mathcal{F}(A)$  is associative.
- (ii)  $G_2$  is satisfied for multiplication in  $\mathcal{F}(A)$ , because the constant function  $C : A \rightarrow R$  such that  $x \mapsto 1$  acts as the identity in  $\mathcal{F}(A)$ . For we have

$$(Cf)(x) = C(x)f(x) = 1f(x) = f(x) \quad \text{for all } x \in A.$$

Hence, by Definition 1.7.1,  $Cf = f$  for all  $f$  in  $\mathcal{F}(A)$ .

- (iii) Multiplication in  $\mathcal{F}(A)$  is commutative.
- (iv) Multiplication is distributive over addition in  $\mathcal{F}(A)$ .

But  $\mathcal{F}(A)$  for pointwise multiplication is not a group, because  $G_3$  is not satisfied as we shall now show.

Suppose  $f : A \rightarrow R$  is a function such that  $f(x_0) = 0$  for some  $x_0 \in A$ . In order to get a function  $g : A \rightarrow R$  such that  $fg = C$ , we must have

$$(fg)(x) = C(x) \quad \text{for all } x \in A$$

or 
$$f(x)g(x) = C(x) = 1 \quad \text{for all } x \in A.$$

But at  $x = x_0$ , this means  $0g(x_0) = 1$ . No real number  $g(x_0)$  can satisfy this requirement. Therefore, no  $g$  exists with the property  $fg = C$ . This means there are functions  $f$  in  $\mathcal{F}(A)$  which have no inverse for multiplication.

If we look at all the properties of addition and multiplication which we have listed, we find that  $\mathcal{F}(A)$  is a commutative ring with unity for these two operations.

The question whether  $\mathcal{F}(A)$  is a group for scalar multiplication does not arise, because scalar multiplication is not a binary operation. But scalar multiplication does satisfy certain properties, which we now list.

Given  $f, g \in \mathcal{F}(A)$ , and  $\lambda, \mu \in R$ , we have

- (i)  $\lambda(f + g) = \lambda f + \lambda g$ ,
- (ii)  $(\lambda + \mu)f = \lambda f + \mu f$ ,
- (iii)  $\lambda(\mu f) = (\lambda\mu)f = \mu(\lambda f)$ ,
- (iv)  $\lambda(fg) = (\lambda f)g = f(\lambda g)$ ,
- (v)  $1f = f$ .

The reader should check these carefully. Just for guidance we check (ii), namely,

$$(\lambda + \mu)f = \lambda f + \mu f.$$

Since this is an equality between functions, we have to prove

$$((\lambda + \mu)f)(x) = (\lambda f + \mu f)(x) \text{ for all } x \in A.$$

$$\text{The left-hand side} = ((\lambda + \mu)f)(x)$$

$$= (\lambda + \mu)(f(x)) \quad (\text{by Definition 1.7.4})$$

$$= \lambda f(x) + \mu f(x) \quad (\text{by F9 since } R \text{ is a field})$$

$$= (\lambda f)(x) + (\mu f)(x) \quad (\text{by Definition 1.7.4})$$

$$= (\lambda f + \mu f)(x) \quad (\text{by Definition 1.7.3}).$$

**1.7.7 Remark** By now it must be clear to the reader that addition, multiplication, and scalar multiplication of functions are not the same as addition, etc., of ordinary numbers, though they may appear to be so. He should be careful to understand, at any stage of the working, whether the symbol that is being used is just a number or a function.

All the results in this article are also true for  $\mathcal{F}_C(A)$ , i.e. the set of all complex valued functions on  $A$ . The only precaution we have to take is that  $\lambda$ , instead of being just a real number, can be any complex number.

We shall end this article by introducing an important concept, called *inverse function*.

**1.7.8 Definition** Let  $f: A \rightarrow B$  be a one-one and onto function. Then the *inverse* of  $f$ , written as  $f^{-1}$ , is the function

$$f^{-1}: B \rightarrow A$$

defined by



$f^{-1}(y) =$  that unique  $x \in A$  for which  $f(x) = y$ .

Note that Definition 1.7.8 is valid, because by Remark 1.4.9 it follows that, since  $f$  is one-one and onto, there is a one-one correspondence between  $A$  and  $B$ .

**Example 1.59** Let  $f: R \rightarrow R$  be defined by  $x \mapsto x + 1$ . Then  $f^{-1}: R \rightarrow R$  is given by  $f^{-1}(y) = y - 1$  for all  $y \in R$ .

If the function  $f: A \rightarrow B$  is not one-one or not onto, the inverse does not exist as a function from  $B$ .

If the function  $f: A \rightarrow B$  is one-one but not onto, the inverse does not exist as a function from  $B$ . But the situation can be retrieved by modifying the function  $f$ . Write  $f_1: A \rightarrow R(f)$  by defining  $f_1(x) = f(x)$  for all  $x \in A$ . Essentially,  $f_1$  is the same as  $f$ . The only difference between  $f$  and  $f_1$  is that the target set  $B$  of  $f$  has been restricted to  $R(f) \subset B$  without any damage.

Obviously,  $f_1$  is one-one and onto; so  $f_1$  has an inverse. This inverse is defined on  $R(f)$ . Therefore, some authors say (legitimately!) that though the inverse of  $f$  does not exist as a function from  $B$ , it exists as a function from  $R(f)$ , a subset of  $B$ .

We shall now give two important properties. If  $f: A \rightarrow B$  is one-one and onto, then

$$(i) \quad f \circ f^{-1} = I_B \quad \text{and}$$

$$(ii) \quad f^{-1} \circ f = I_A.$$

The proof of (i) is left to the reader. We shall prove (ii) :

$f: A \rightarrow B$  is a one-one and an onto map. Therefore,  $f^{-1}: B \rightarrow A$  exists. The composition  $f^{-1} \circ f$  is a map from  $A$  to  $A$  such that

$$\begin{aligned} (f^{-1} \circ f)(x) &= f^{-1}(f(x)) && \text{(by Definition 1.7.2)} \\ &= f^{-1}(y) && \text{(by writing } y = f(x)) \\ &= x && \text{(by Definition 1.7.8)} \end{aligned}$$

for all  $x \in A$ . Hence,  $f^{-1} \circ f = I_A$ .

As mentioned in Remark 1.4.3, the symbol  $f^{-1}$  is also used in certain contexts in mathematics for denoting the pre-image under  $f$ . But we shall not use  $f^{-1}$  in that context. We shall use  $f^{-1}$  only when it is the inverse of  $f$ .

### Problem Set 1.7

1. Prove that the composition of functions is associative, i.e. for any three functions  $h: A \rightarrow B$ ,  $g: B \rightarrow C$ , and  $f: C \rightarrow D$ , we have

$$(f \circ g) \circ h = f \circ (g \circ h).$$

2. Let  
and

$$\begin{aligned} f(x) &= \sin x, \quad -\pi < x \leq \pi \\ g(x) &= x^2, \quad -\infty < x < \infty. \end{aligned}$$

Describe  $g \circ f$ . Is  $f \circ g$  defined? Justify your answer. Change the domain of  $g$  so that  $f \circ g$  can be defined. Then describe  $f \circ g$ .

3. Repeat Problem 2 for  $f(x) = 1 + x^2$ ,  $-1 \leq x \leq 1$ , and  $g(x) = \sqrt{x}$ ,  $x \geq 0$ .
4. Prove that the set of all real-(complex) valued continuous functions defined on  $[0, 1]$  is a group under pointwise addition of functions.
5. Prove that the set of all real-valued differentiable functions on  $(a, b)$  is a group under pointwise addition.
6. If  $f$  and  $g$  are defined as in Problem 2, describe  $f + g$ ,  $f - g$ , and  $fg$  for  $-\pi \leq x \leq \pi$ .
7. If  $f$  and  $g$  are defined as in Problem 3, describe  $f + g$ ,  $f - g$  and  $fg$  for  $0 \leq x \leq 1$ .
8. In  $R^2$  define equality, addition, and scalar multiplication as follows :  
*Equality*  $(x_1, x_2) = (y_1, y_2)$  if  $x_1 = y_1, x_2 = y_2$  ;  
*Addition*  $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$  ;  
*Scalar multiplication*  $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$ , where  $\alpha$  is any real number.

A function  $f: R^2 \rightarrow R^2$  is said to be *linear* if it satisfies the following properties :

- (i)  $f((x_1, x_2) + (y_1, y_2)) = f(x_1, x_2) + f(y_1, y_2)$
- (ii)  $f(\alpha(x_1, x_2)) = \alpha f(x_1, x_2)$ ,  
for all real  $\alpha$ .

Prove that each function  $f: R^2 \rightarrow R^2$  defined below is linear.

- (a)  $f(x_1, x_2) = (x_2, x_1)$
- (b)  $f(x_1, x_2) = (x_1, -x_2)$
- (c)  $f(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ .

9. Prove that for the linear function  $f: R^2 \rightarrow R^2$  defined in Problem 8, we have

$$f(0, 0) = (0, 0).$$

10. Consider  $\mathcal{F}(G, H)$ , the set of all functions from a group  $G$  to a non-abelian group  $H$  with a binary operation  $\cdot$ . For  $f, g \in \mathcal{F}(G, H)$ , define  $fg$  as  $(fg)(x) = f(x) \cdot g(x)$ . Is  $fg = gf$ ?

11. True or false?

- (a) The function  $f: C \rightarrow R$  defined by  $f(a + ib) = a$  is an onto function.
- (b) The function  $f: C \rightarrow R$  defined by  $f(a + ib) = b$  is a one-one function.

- (c) The function  $f : C \rightarrow R \times R$  defined by  $f(a + ib) = (a, b)$  is a one-one onto function.
- (d) It is possible to define multiplication of functions, which is noncommutative.

We conclude this article by drawing the reader's attention to the two ideas in this chapter that will form the central theme of this book. They are 'addition' and 'scalar multiplication' of functions. The study of these two concepts in depth and in more general situations is the study of linear algebra.

# Vectors

In this chapter we shall study vectors in a plane and in space. These concepts originally arose in mechanics. A vector in mechanics stands for a line segment directed from a point  $P$  (called the *initial point*) to a point  $Q$  (called the *terminal point*). It is denoted by the symbol  $\overrightarrow{PQ}$ , where the arrow indicates that it is a vector. The addition of vectors is done by the familiar law of parallelogram of forces. The negative of  $\overrightarrow{PQ}$  is denoted by  $\overrightarrow{QP}$ . From this the rule of subtraction follows. The study of vectors starting from this approach is called the geometric approach.

The alternative approach, called the algebraic approach, is more relevant to the subject matter of this book. Whereas the geometric approach leans heavily on the fundamental geometric concepts of 'length' and 'angle', the algebraic approach uses only the properties of  $R$ , the set of real numbers.

Today we find that the use of vectors transcends its original geometrical background of physical space and finds its application in various fields, including social sciences. For this reason, our study of vectors will be mainly algebraic. The geometric aspect will, however, be presented side by side, in order to supplement the understanding. In fact, keeping track of the geometrical perspective is very useful in that it provides a deeper insight into the study of linear algebra. At the end of this chapter some elementary applications of vectors to geometry are also presented.

We are familiar with the cartesian coordinate system in a plane. Before studying vectors we shall introduce the cartesian coordinate system in space.

## 2.1 SPACE COORDINATES

We identify a point in a plane with the help of two axes, called the  $x$ -axis and the  $y$ -axis. To locate a point in space, we need three mutually perpendicular lines  $X'OX$ ,  $Y'OY$ , and  $Z'OZ$  through a point  $O$  (Figure 2.1).

This point  $O$  is called the origin and the lines are called  $x$ -axis,  $y$ -axis, and  $z$ -axis, respectively. These three mutually perpendicular lines, called the

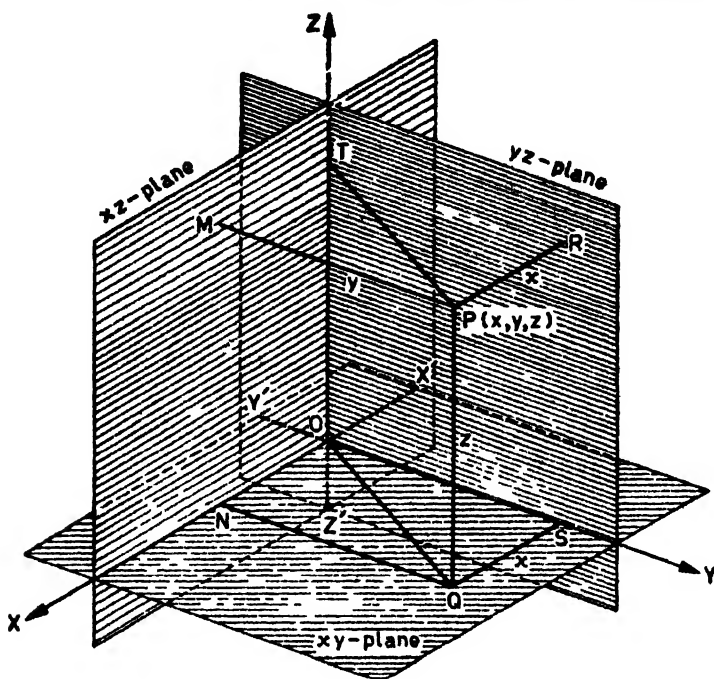


FIGURE 2.1

coordinate axes, determine three mutually perpendicular coordinate planes  $XOY$ ,  $YOZ$ , and  $ZOX$ , called the  $xy$ -plane, the  $yz$ -plane, and the  $xz$ -plane, respectively. These three planes divide space into eight parts, called octants. Note that

- $x$ -axis is the intersection of the  $xy$ -plane and the  $xz$ -plane;
- $y$ -axis is the intersection of the  $xy$ -plane and the  $yz$ -plane; and
- $z$ -axis is the intersection of the  $yz$ -plane and the  $xz$ -plane.

We choose the positive directions  $OX$ ,  $OY$ , and  $OZ$  of these axes in such a way that, if a right-handed screw placed at  $O$  is rotated from  $OX$  to  $OY$  through  $90^\circ$ , it moves in the direction of  $OZ$ . The axes so chosen are said to form a *right-handed system*. Figure 2.2 shows a right-handed system and Figure 2.3 shows a *left-handed system*.

Let  $P$  be a point in space (see Figure 2.1). Then the distance of  $P$  from the  $xy$ -plane is called the  $z$ -coordinate of point  $P$ . Let  $Q$  be the foot of the perpendicular from  $P$  on the  $xy$ -plane. Then the distance of  $Q$  from the  $y$ -axis is called the  $x$ -coordinate of point  $P$  and that from the  $x$ -axis is the  $y$ -coordinate of point  $P$ . Obviously, the  $x$ -coordinate of point  $P$  is its

distance from the  $yz$ -plane and the  $y$ -coordinate is its distance from the  $zx$ -plane.

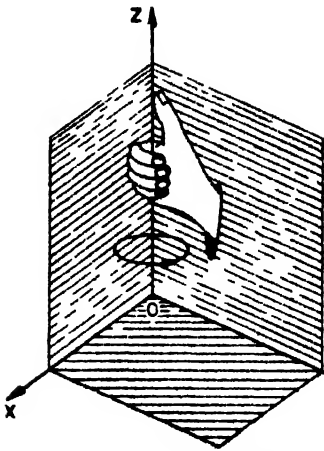


FIGURE 2.2

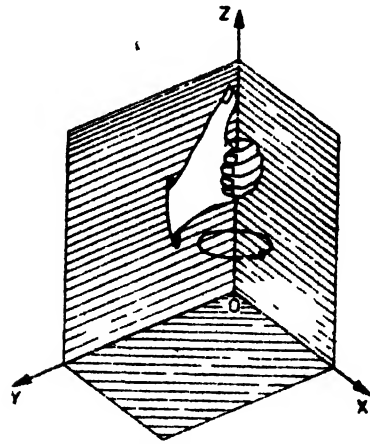


FIGURE 2.3

If  $x$ ,  $y$ , and  $z$  are respectively the  $x$ -coordinate,  $y$ -coordinate, and  $z$ -coordinate of point  $P$ , we use the ordered triple  $(x, y, z)$  to denote  $P$ . It

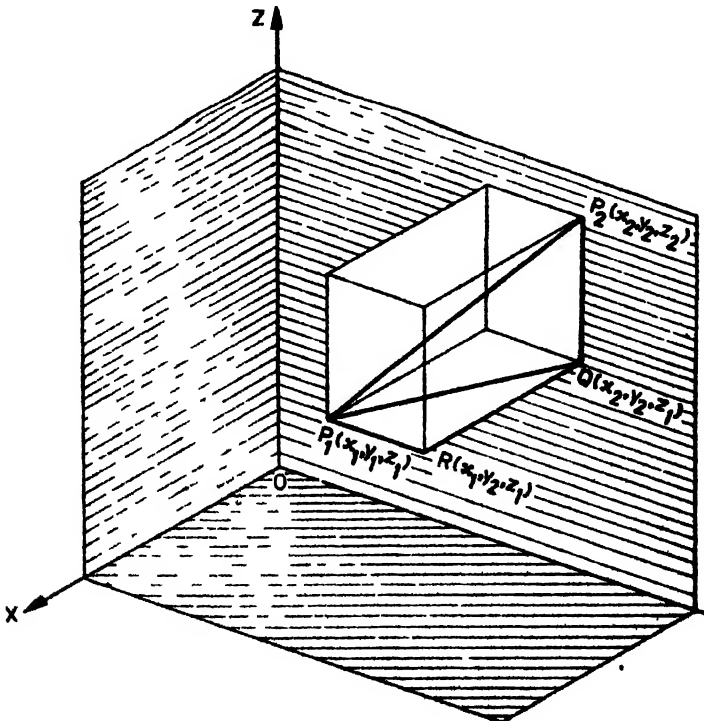


FIGURE 2.4

is customary to say that the coordinates of  $P$  relative to the origin  $O$  are  $(x, y, z)$ . Thus, corresponding to each point in space, there exists a unique ordered triple of real numbers which gives the distance of  $P(x, y, z)$  from the three coordinate planes. Similarly, corresponding to each ordered triple of real numbers, there exists a unique point in space, which is situated at a distance of  $x$  units from the  $yz$ -plane,  $y$  units from the  $zx$ -plane, and  $z$  units from the  $xy$ -plane. Obviously, the coordinates of the origin are  $(0, 0, 0)$ .

*Distance between the points  $P_1(x_1, y_1, z_1)$  and  $P_2(x_2, y_2, z_2)$*

To find the distance between the points  $P_1$  and  $P_2$ , draw  $P_2Q$  perpendicular from  $P_2$  to the plane  $P_1RQ$  parallel to the  $xy$ -plane and passing through  $P_1$ , as shown in Figure 2.4. Further, draw  $QR$  and  $P_1R$  parallel to the  $x$ -axis and  $y$ -axis, respectively. Then

$$\begin{aligned} P_1P_2 &= \sqrt{(P_1Q)^2 + (P_2Q)^2} \\ &= \sqrt{(RQ)^2 + (P_1R)^2 + (P_2Q)^2} \\ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \quad (1) \end{aligned}$$

**Example 2.1** The distance between the points  $P(1, -1, 3)$  and  $Q(2, 1, -7)$  is  $\sqrt{(-10)^2 + (2^2) + (1)^2} = \sqrt{105}$ .

### Problem Set 2.1

- Locate the points :  
 (a)  $(1, 1, 1)$       (b)  $(1, 1, 2)$       (c)  $(-1, 3, -2)$   
 (d)  $(-1, -2, -3)$ .
- Describe the loci :  
 (a)  $y = \text{constant}$       (b)  $z = \text{constant}$   
 (c)  $x = 1, y = 2z$       (d)  $y = -5, z^2 = 4x$   
 (e)  $y^2z^4 = 1, x = 2$       (f)  $x \leq 0$   
 (g)  $x \geq y$       (h)  $y \geq 2z, x = 3$ .
- Find the length of the segment  $AB$ , with end points :  
 (a)  $A(0, 0, 0), B(1, 1, 1)$   
 (b)  $A(1, 2, -1), B(0, -1, 1)$   
 (c)  $A(3, 2, -1), B(3, 1, 2)$   
 (d)  $A(-1, 3, -2), B(1, 2, 0)$ .
- Look at the rectangular parallelepiped in Figure 2.5. If its length  $AB = 3.5$  units, breadth  $BC = 2$  units, and height  $BD = 1.5$  units, and the coordinates of  $A$  are  $(1, 2, 3)$ , find the coordinates of all its vertices and the length of its diagonals.

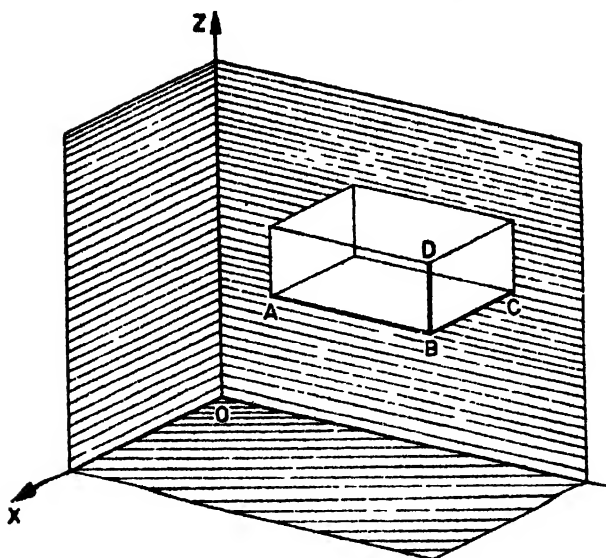


FIGURE 2.5

## 2.2 VECTORS—ADDITION AND SCALAR MULTIPLICATION

We start with the geometrical concept of a vector. A vector is a directed line segment  $\overrightarrow{PQ}$  with an initial point  $P$  and a terminal point  $Q$ . In Figure 2.6  $\overrightarrow{PQ}$  and  $\overrightarrow{RS}$  are two vectors. The *length* or *magnitude* of vector

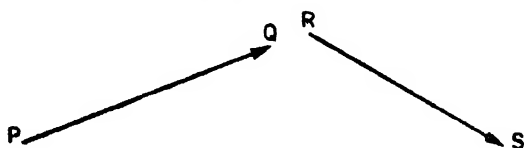


FIGURE 2.6

$PQ$  is the distance between  $P$  and  $Q$  and is denoted by  $|\overrightarrow{PQ}|$ . A vector of unit length is called a *unit vector*.

Two vectors  $\overrightarrow{OP}$  and  $\overrightarrow{O'P'}$  are said to be *parallel* if the line segments  $OP$  and  $O'P'$  are parallel. Two parallel vectors may have the same or opposite directions. In geometry two parallel vectors are considered to be equivalent if they have the same length and the same direction. In Figure 2.7  $\overrightarrow{OP}$ ,  $\overrightarrow{O'P'}$ , and  $\overrightarrow{RS}$  are parallel vectors, and  $\overrightarrow{OP}$  and  $\overrightarrow{RS}$  are equivalent vectors. However, in mechanics the need may arise to consider



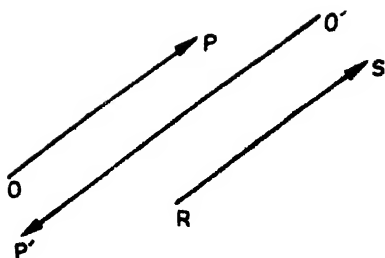


FIGURE 2.7

them as different. For example,  $\vec{OP}$  and  $\vec{RS}$  may represent two forces acting at two different points  $O$  and  $R$ . In our study of vectors we shall ignore this distinction between initial points. We shall assume that there is always a coordinate system with an origin  $O$ . Given any vector  $\vec{AB}$ , unless otherwise stated, we shall always think of a parallel vector  $\vec{OP}$  with an initial point  $O$  and a terminal point  $P$  such that  $\vec{OP}$  and  $\vec{AB}$  have the same directions and  $|\vec{OP}| = |\vec{AB}|$ , as in Figure 2.8.

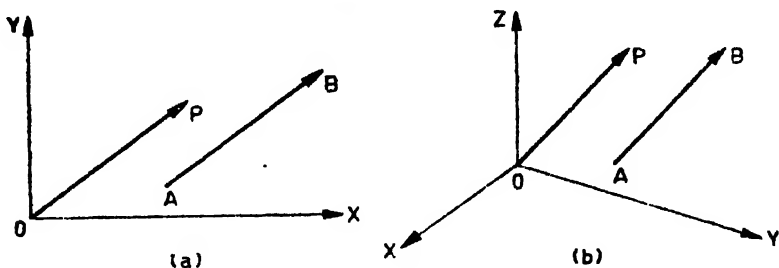


FIGURE 2.8

Note that, given  $\vec{AB}$ , we arrive at only one  $\vec{OP}$  in this way. Thus, all our vectors shall start from a fixed point, namely, the origin, in the chosen coordinate system. Such vectors are also called *free vectors*.

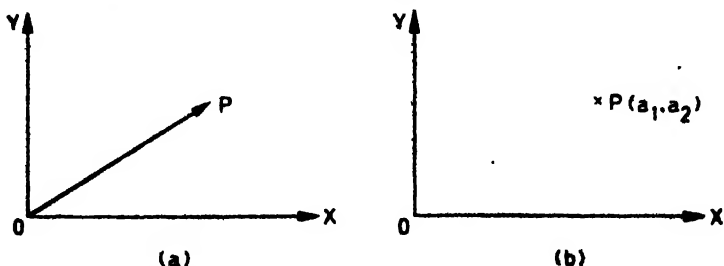


FIGURE 2.9

Consider the vector  $\overrightarrow{OP}$  in a plane, where  $P$  is the point with coordinates  $(a_1, a_2)$  relative to the chosen coordinate system referred to  $O$  as origin (Figure 2.9).

The sum of two vectors  $\overrightarrow{OP}$  and  $\overrightarrow{OQ}$  is given by  $\overrightarrow{OR}$ , the diagonal of the parallelogram whose adjacent sides are  $OP$  and  $OQ$  (Figure 2.10). If  $P$  and  $Q$  have coordinates  $(a_1, a_2)$  and  $(b_1, b_2)$ , respectively, then elementary geometry tells us that the coordinates of  $R$  are  $(a_1 + b_1, a_2 + b_2)$ . (Why?) Suppressing the directed line segments and writing only the coordinates of the vertices, we have Figure 2.10 (b).

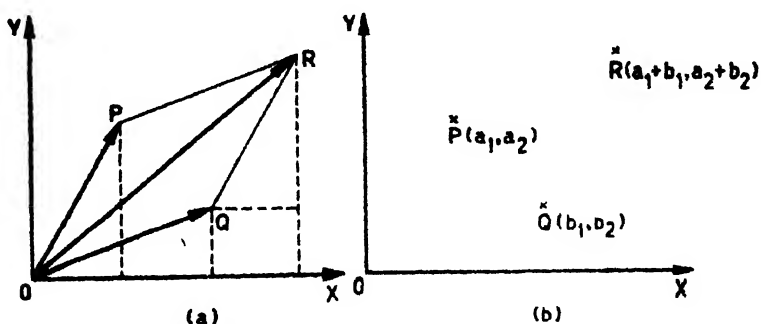


FIGURE 2.10

Geometrically,  $\lambda \overrightarrow{OP}$ , with  $\lambda > 0$ , is defined as the vector whose length is  $\lambda$ -times the length of  $OP$ , i.e.  $\lambda |\overrightarrow{OP}|$ , and whose direction is the same as that of  $\overrightarrow{OP}$ . If  $\lambda < 0$ , then  $\lambda \overrightarrow{OP}$  denotes a vector of length  $-\lambda |\overrightarrow{OP}|$  in a direction opposite to that of  $\overrightarrow{OP}$ . As before we get the pairs of diagrams as shown in Figures 2.11 (a) and (b), and 2.12 (a) and (b). (In these diagrams  $|\lambda| > 1$ .)

An analogous situation arises in space, where the point  $P$  has the coordinates  $(a_1, a_2, a_3)$ .

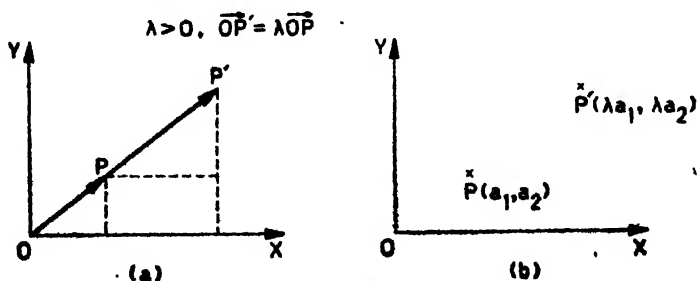


FIGURE 2.11

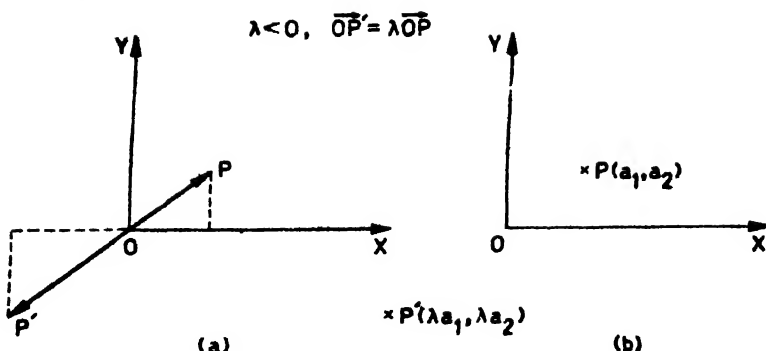


FIGURE 2.12

If we carefully analyse the transition from (a) to (b) in Figures 2.10, 2.11, and 2.12, we observe that the coordinates  $a_1$  and  $a_2$  in the plane (and the coordinates  $a_1$ ,  $a_2$ , and  $a_3$  in space) of the point  $P$  perhaps contain everything that we want of the vector  $\vec{OP}$ . It is this observation that leads to the following algebraic definition of a vector and to the succeeding definitions in this chapter.

**2.2.1 Definition** (a) A *plane vector* is an ordered pair  $(a_1, a_2)$  of real numbers.

(b) A *space vector* is an ordered triple  $(a_1, a_2, a_3)$  of real numbers.

We shall *not* make any distinction between the plane vector  $(a_1, a_2)$  and the directed line segment  $\vec{OP}$ , where  $O$  is the origin and  $P$  is the point whose cartesian coordinates are  $(a_1, a_2)$ . In fact, we shall very often write

$$u = (a_1, a_2) = \vec{OP}.$$

In this case the vector  $(a_1, a_2)$  is also called the *position vector* of  $P$ . The vector  $(0, 0)$  is called the *zero vector* in a plane. Similarly, in the case of space vectors, we write  $u = (a_1, a_2, a_3) = \vec{OP}$ . The vector  $(0, 0, 0)$  is called the *zero vector* in space.

The length or magnitude of the vector  $u = (a_1, a_2) = \vec{OP}$  is the length of the segment  $OP$ . It is  $\sqrt{(a_1^2 + a_2^2)}$ . In the case of the space vector  $u = (a_1, a_2, a_3) = \vec{OP}$ , we have  $|u| = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$ .

**Example 2.2** The length of the vector  $(3, 4)$  is  $\sqrt{(9 + 16)} = 5$ .

**Example 2.3** The length of the vector  $(1, 3, -1)$  is  $\sqrt{(1 + 9 + 1)} = \sqrt{11}$ .

**Example 2.4** The vector  $(\sqrt{2}/10, -7\sqrt{2}/10)$  is a unit vector, since its length is 1.

**Example 2.5** The vector  $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$  is a unit vector, since its length is 1.

**2.2.2 Definition** (a) The set of all plane vectors (i.e. the set of all ordered pairs of real numbers) is denoted by  $V_2$ .

(b) The set of all space vectors (i.e. the set of all ordered triples of real numbers) is denoted by  $V_3$ .

Note that  $V_2$  is the cartesian product  $R \times R$ . For this reason  $V_2$  may also be denoted by  $R^2$ . Similarly,  $V_3 = R \times R \times R = R^3$ .

**2.2.3 Definition (Equality)** (a) Two plane vectors  $(a_1, a_2)$  and  $(b_1, b_2)$  are said to be equal if  $a_1 = b_1$  and  $a_2 = b_2$ , i.e.  $(a_1, a_2) = (b_1, b_2)$  if  $a_i = b_i, i = 1, 2$ .

(b) Two space vectors  $(a_1, a_2, a_3)$  and  $(b_1, b_2, b_3)$  are said to be equal if  $a_1 = b_1, a_2 = b_2$ , and  $a_3 = b_3$ , i.e.  $(a_1, a_2, a_3) = (b_1, b_2, b_3)$  if  $a_i = b_i, i = 1, 2, 3$ .

The *direction* of a nonzero vector in  $V_2$  is the radian measure  $\theta$ ,  $0 \leq \theta < 2\pi$ , of the angle from the positive direction of the  $x$ -axis to the vector  $\overrightarrow{OP}$  measured counter-clockwise. If  $u = (a_1, a_2) = \overrightarrow{OP}$ , then the direction  $\theta$  of  $u$  is given by

$$\sin \theta = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \quad \cos \theta = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}$$

In  $V_3$  the direction of a vector is given by its direction cosines, which we shall study in § 2.4. Note that the direction of the zero vector is undetermined.

**Example 2.6** The direction  $\theta$  of the vector  $(1/\sqrt{2}, 1/\sqrt{2})$  is given by  $\sin \theta = 1/\sqrt{2} = \cos \theta$ . So  $\theta = \pi/4$ .

**Example 2.7** The direction of the vector  $(3, 4)$  is given by  $\sin \theta = 4/5$ ,  $\cos \theta = 3/5$ .

Now we shall define the addition of two vectors in a plane as well as in space. These definitions correspond to the geometrical definition of addition.

**2.2.4 Definition (Addition)** (a) Addition of vectors in  $V_2$  is defined as

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$$

for all vectors  $(a_1, a_2), (b_1, b_2)$  in  $V_2$ .

(b) Addition of vectors in  $V_3$  is defined as

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1 + b_1, a_2 + b_2, a_3 + b_3),$$

for all vectors  $(a_1, a_2, a_3), (b_1, b_2, b_3)$  in  $V_3$ .

**Example 2.8** The sum of the vectors  $(3, -1)$  and  $(-1, 4)$  is the vector  $(3, -1) + (-1, 4) = (2, 3)$ .

**Example 2.9** The sum of the vectors  $(-1, 0, 7)$  and  $(3, 2, 1)$  is the vector  $(-1, 0, 7) + (3, 2, 1) = (2, 2, 8)$ .

Obviously, these operations of addition '+' are binary operations in  $V_2$  and  $V_3$ , respectively. We shall now prove that  $V_2$  and  $V_3$  are commutative groups for these operations.

**2.2.5 Theorem**  $V_2(V_3)$ , the set of all plane (space) vectors, under the operation of addition, as in Definition 2.2.4, is a commutative group.

*Proof:* The proof for  $V_3$  is left to the reader. To prove that  $V_2$  is a commutative group, we check the following :

**G1** Vector addition is associative

Let  $u = (a_1, a_2)$ ,  $v = (b_1, b_2)$ , and  $w = (c_1, c_2)$  be three vectors in  $V_2$ . Then

$$(u + v) + w = ((a_1 + b_1) + c_1, (a_2 + b_2) + c_2)$$

$$\text{and } u + (v + w) = (a_1 + (b_1 + c_1), a_2 + (b_2 + c_2)).$$

But we know that for real numbers  $a_i, b_i$ , and  $c_i$  (cf § 1.6)

$$(a_i + b_i) + c_i = a_i + (b_i + c_i), \quad i = 1, 2.$$

Therefore,  $(u + v) + w = u + (v + w)$  for all  $u, v, w \in V_2$ .

**G2** Existence of identity

For any  $u = (a_1, a_2)$  in  $V_2$ , we have

$$(0, 0) + (a_1, a_2) = (a_1, a_2) = (a_1, a_2) + (0, 0).$$

So the zero vector  $(0, 0)$  plays the role of identity. The zero vector usually denoted by  $0$ .

**G3** Existence of inverse

Given a vector  $u = (a_1, a_2)$ , the vector  $x = (-a_1, -a_2)$  satisfies

$$x + u = 0 = u + x.$$

This vector  $x$  is called the *additive inverse* of  $u$  or the *negative* of  $u$  and is denoted by  $-u$ .

**G4** Vector addition is commutative

Let  $u = (a_1, a_2)$  and  $v = (b_1, b_2)$  be two vectors in  $V_2$ . Then

$$u + v = (a_1 + b_1, a_2 + b_2)$$

$$\text{and } v + u = (b_1 + a_1, b_2 + a_2).$$

But we know that for real numbers  $a_1, a_2, b_1, b_2$  (cf § 1.6)

$$a_i + b_i = b_i + a_i, \quad i = 1, 2.$$

Therefore,  $u + v = v + u$  for all  $u, v \in V_2$ .

This completes the proof that  $V_2$ , under the operation '+', is a commutative group. ■

It may be noted that the additive identity of the group  $V_2$  is the vector  $(0, 0, 0)$ . We shall use the same symbol  $0$  for this vector also. In  $V_3$  if  $u = (a_1, a_2, a_3)$ , then  $-u = -(a_1, a_2, a_3) = (-a_1, -a_2, -a_3)$ .

**2.2.6 Definition (Scalar multiplication)** Multiplication of vectors in  $V_2$  by

a real number is defined as

$$\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2)$$

for every  $(a_1, a_2) \in V_2$  and every real number  $\lambda$ . Similarly, in  $V_3$ ,

$$\lambda(a_1, a_2, a_3) = (\lambda a_1, \lambda a_2, \lambda a_3)$$

for every  $(a_1, a_2, a_3) \in V_3$  and every real number  $\lambda$ .

**Example 2.10** If the vector  $u = (3, -1)$ , then the vector  $2u = (6, -2)$  and  $\frac{1}{3}u = (1, -1/3)$ .

Note that scalar multiplication is *not* a binary operation. (Why?) However, it does satisfy certain natural properties which we shall list and prove.

### 2.2.7 Properties of Scalar Multiplication

For all vectors  $u, v$  in  $V_2$  (or in  $V_3$ ) and real numbers  $\alpha, \beta$ , we have

$$(a) \quad \alpha(u + v) = \alpha u + \alpha v.$$

$$(b) \quad (\alpha + \beta)u = \alpha u + \beta u.$$

$$(c) \quad \alpha(\beta u) = (\alpha\beta)u = \beta(\alpha u).$$

$$(d) \quad 1u = u.$$

$$(e) \quad 0u = 0.$$

*Proof:* We shall prove (a) as an illustration of the method of proving these properties. Further, we shall prove this in  $V_3$  only. The rest are left to the reader.

Let  $u = (a_1, a_2, a_3)$  and  $v = (b_1, b_2, b_3)$  be two vectors in  $V_3$ , and  $\alpha$  be a real number. Then

$$u + v = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$\begin{aligned} \text{and} \quad \alpha(u + v) &= (\alpha(a_1 + b_1), \alpha(a_2 + b_2), \alpha(a_3 + b_3)) \\ &= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3) \end{aligned} \quad (1)$$

by properties of real numbers (cf § 1.6). Further, we have

$$\begin{aligned} \alpha u + \alpha v &= \alpha(a_1, a_2, a_3) + \alpha(b_1, b_2, b_3) \\ &= (\alpha a_1, \alpha a_2, \alpha a_3) + (\alpha b_1, \alpha b_2, \alpha b_3) \\ &= (\alpha a_1 + \alpha b_1, \alpha a_2 + \alpha b_2, \alpha a_3 + \alpha b_3) \\ &= \alpha(u + v) \quad (\text{by (1)}). \blacksquare \end{aligned}$$

The last two properties, namely, (d) and (e), may look trivial to the reader. In fact, they just say

$$\begin{aligned} 1(a_1, a_2, a_3) &= (1a_1, 1a_2, 1a_3) && (\text{Definition 2.2.6}) \\ &= (a_1, a_2, a_3) \end{aligned}$$

and

$$\begin{aligned} 0(a_1, a_2, a_3) &= (0a_1, 0a_2, 0a_3) && (\text{Definition 2.2.6}) \\ &= (0, 0, 0). \end{aligned}$$

Let the reader reserve his opinion until he comes to the discussion of an abstract vector space in Chapter 3. Then he will see the significance of properties (d) and (e) and also the delicate difference between them.

**2.2.8 Definition** The difference of two vectors  $u$  and  $v$ , written as  $u - v$ , is defined by  $u - v = u + (-v)$ .

*Example 2.11* Let  $u = (3, 4)$ ,  $v = (-1, 3)$ . Then  $u - v = u + (-v) = (3, 4) + (1, -3) = (4, 1)$ .

We shall now find the vector  $u - v$  geometrically. Let  $u = \vec{OP} = (a_1, a_2)$  and  $v = \vec{OQ} = (b_1, b_2)$ . The negative of  $v$ , i.e.  $-v$ , is  $(-b_1, -b_2)$ . Geometrically, it is the vector  $\vec{QO}$  which is the same as  $\vec{OQ'}$  (see Figure 2.13), where  $Q'$  is the point whose cartesian coordinates

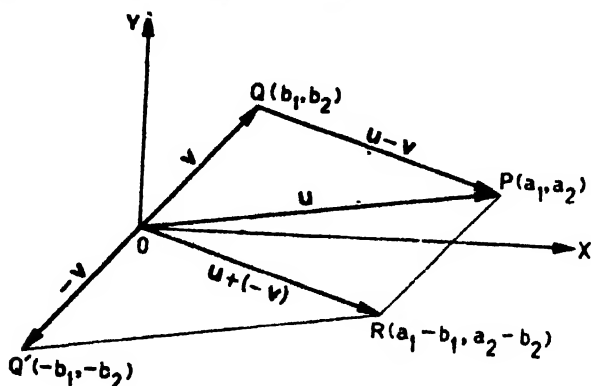


FIGURE 2.13

are  $(-b_1, -b_2)$ . The vector  $u - v$  is the diagonal of the parallelogram formed by  $u$  and  $v$ . It is the vector  $\vec{OR} = \vec{QP} = (a_1 - b_1, a_2 - b_2)$ . Analogously, we can find  $u - v$  in  $V_3$ .

For every nonzero vector  $u$ , the vector  $\frac{1}{|u|} u$  is a unit vector in the direction of  $u$ . If  $u = (a_1, a_2) \neq 0$ , the unit vector in the direction of  $u$  is

$$\left( \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2}} \right).$$

If  $u = (a_1, a_2, a_3) \neq 0$ , the unit vector in the direction of  $u$  is

$$\left( \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}} \right).$$

*Example 2.12* If  $u = (3, 4)$ , the unit vector in the direction of  $u$  is  $(3/5, 4/5)$ .

*Example 2.13* If  $u = (1, 1, 1)$ , the unit vector in the direction of  $u$  is  $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ .

In a plane the unit vector in the direction of an angle  $\theta$  is  $(\cos \theta, \sin \theta)$ , as shown in Figure 2.14. It is also called the unit vector at an angle  $\theta$  with

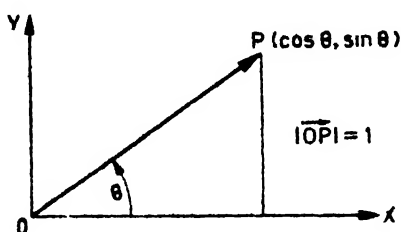


FIGURE 2.14

the positive direction of the  $x$ -axis. The unit vectors in the positive directions of the  $x$ -axis and the  $y$ -axis are  $(1, 0)$  and  $(0, 1)$ , respectively. They are denoted by  $i$  and  $j$ , respectively, i.e.  $i = (1, 0)$  and  $j = (0, 1)$ , as in Figure 2.15.

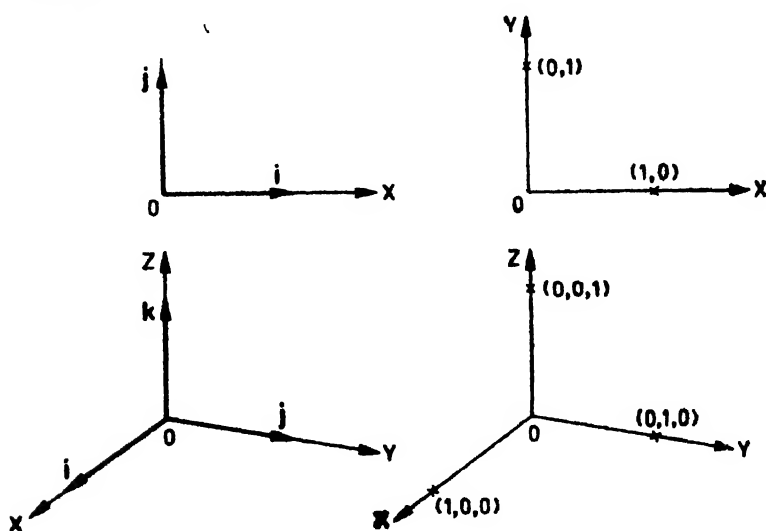


FIGURE 2.15

In space the unit vectors along the positive directions of the  $x$ -axis, the  $y$ -axis, and the  $z$ -axis are respectively  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ . They are denoted by  $i$ ,  $j$ , and  $k$ , respectively, i.e.  $i = (1, 0, 0)$ ,  $j = (0, 1, 0)$ , and  $k = (0, 0, 1)$ . Though we use the same letters  $i$  and  $j$  here as in the case of plane vectors, there cannot be any confusion, because the context will always show whether we are talking of plane vectors or space vectors. The importance of unit vectors is brought out by the following theorem.



**2.2.9 Theorem** (a) Every plane vector is of the form  $a_1i + a_2j$  and every vector of this form is a plane vector.

(b) Every space vector is of the form  $a_1i + a_2j + a_3k$  and every vector of this form is a space vector.

**Proof:** We shall prove (b). The proof of (a) is analogous. Let  $(a_1, a_2, a_3)$  be a vector in  $V_3$ . Then

$$\begin{aligned}(a_1, a_2, a_3) &= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) \\ &= a_1(1, 0, 0) + a_2(0, 1, 0) + a_3(0, 0, 1) \\ &= a_1i + a_2j + a_3k.\end{aligned}$$

Thus, every vector in  $V_3$  is of the form  $a_1i + a_2j + a_3k$ . On the other hand, a vector of the form  $a_1i + a_2j + a_3k$  is the sum of vectors  $a_1i$ ,  $a_2j$ , and  $a_3k$  of  $V_3$ . Since vector addition is an associative binary operation,  $a_1i + a_2j + a_3k$  is a vector in  $V_3$ , i.e. a space vector. ■

Geometrically, Theorem 2.2.9 means that every vector can be represented as a sum of scalar multiples of unit vectors along the coordinate axes, as shown in Figure 2.16.

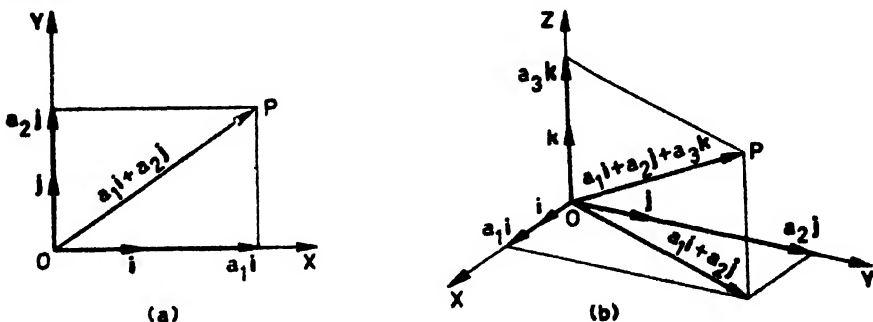


FIGURE 2.16

The numbers  $a_1, a_2$  are called the *components* of the plane vector  $u = (a_1, a_2) = a_1i + a_2j$ .  $a_1$  is called the  $i$ -component (or  $x$ -component) and  $a_2$  is called the  $j$ -component (or  $y$ -component). We also say that  $a_1$  and  $a_2$  are the coordinates of  $u$  with respect to  $i$  and  $j$ .

Similarly,  $a_1, a_2$ , and  $a_3$  are called the *components* of the space vector  $u = (a_1, a_2, a_3) = a_1i + a_2j + a_3k$ .  $a_1$  is the  $i$ -component,  $a_2$  is the  $j$ -component, and  $a_3$  the  $k$ -component. We also say that  $a_1, a_2$ , and  $a_3$  are coordinates of  $u$  with respect to  $i, j$ , and  $k$ .

### Problem Set 2.2

1. Find the magnitude and direction of the following plane vectors :

- |              |               |               |                  |
|--------------|---------------|---------------|------------------|
| (a) $(2, 3)$ | (b) $(3, -1)$ | (c) $(3, 6)$  | (d) $(-1, 2)$    |
| (e) $i + j$  | (f) $2i - 3j$ | (g) $-i + 4j$ | (h) $-4i - 2j$ . |

2. Find the magnitude of the following space vectors :

- (a)  $(2, -1, 3)$                       (b)  $(3, 0, 4)$   
 (c)  $3i + 2j - k$                       (d)  $-i - 2j + 4k$ .

3. Simplify

- (a)  $3(2, -3)$                       (b)  $4(1, -1, 3)$   
 (c)  $2(1, -1) - 7(5, 1)$                       (d)  $3(2, -1, 4) + 4(-1, 5, 0)$   
 (e)  $7(3i + 2j - k) - 4(2i - j + k)$ .

4. Express the following vectors in terms of the unit vectors  $i, j$ , and  $k$  :

- (a)  $(2, 3)$                       (b)  $(-1, 4)$                       (c)  $(-3, -5)$   
 (d)  $(5, 3, -2)$                       (e)  $(2, 0, 1)$                       (f)  $(-1, 2, 0)$ .

5. Express the following space vectors as sums of scalar multiples of  $u = (1, 1, 1)$ ,  $v = (1, 2, 3)$ , and  $w = (2, 3, 5)$  :

- (a)  $(1, 3, 5)$                       (b)  $(7, -1, 3)$                       (c)  $(2, -2, 1)$   
 (d)  $i + j - k$                       (e)  $-2i + j - 5k$                       (f)  $3i - 7j - 2k$ .

(Hint : Assume each vector to be equal to  $(\alpha u + \beta v + \gamma w)$ , where  $\alpha, \beta$ , and  $\gamma$  are real numbers.)

6. Find a unit vector  $u$  in the direction of each of the vectors in Problems 1 and 2.

7. Find the unit plane vector in the direction :

- (a)  $\theta = -3\pi/4$                       (b)  $\theta = \pi/6$                       (c)  $\theta = 7\pi/4$ .

8. Given a point  $A$ , determine a point  $B$  such that the vector  $\overrightarrow{AB}$  is equal to the vector  $v$  in the following :

- (a)  $v = (1, -2, 3)$ ,  $A(0, 1, -1)$   
 (b)  $v = (3, 1, -2)$ ,  $A(-1, 2, 3)$   
 (c)  $v = (3, -7, 1)$ ,  $A(1, 5, 0)$   
 (d)  $v = (1, -1, 1)$ ,  $A(2, 3, 5)$   
 (e)  $v = (1, -2)$ ,  $A(1, -2)$   
 (f)  $v = (-1, 5)$ ,  $A(7, -3)$ .

9. Given the midpoint of  $AB$ , determine the points  $A$  and  $B$  such that the vector  $\overrightarrow{AB}$  is equal to the vector  $v$  in the following :

- (a)  $v = (1, 3)$ , midpoint of  $AB$  is  $(1, 2)$   
 (b)  $v = (-1, 2)$ , midpoint of  $AB$  is  $(0, 0)$   
 (c)  $v = (3, -1, 2)$ , midpoint of  $AB$  is  $(3, -1, 2)$   
 (d)  $v = (-2, 0, 1)$ , midpoint of  $AB$  is  $(2, 3, -2)$ .

10. Prove that  $V_3$  together with the operation of vector addition is a commutative group.

11. Let  $u = \vec{OA}$  and  $v = \vec{OB}$  be two vectors. Then prove that  $u - v = \vec{BA}$ .
12. Prove the properties of scalar multiplication (b) and (c) stated in § 2.2.7.
13. Prove that  $\alpha u = 0$  iff  $\alpha = 0$  or  $u = 0$ .
14. Let  $u$  be a nonzero vector in  $V_3$ . Then prove that the set  $S = \{\alpha u \mid \alpha \in R\}$  is a group under vector addition.
15. Let  $u$  and  $v$  be two nonzero vectors in  $V_3$ . Then prove that the set  $S = \{\alpha u + \beta v \mid \alpha, \beta \in R\}$  is a group under vector addition.
16. True or false ?
  - (a)  $\frac{1}{2}(2, 0, 4) = (1, 0, 4)$ .
  - (b)  $(a + b, b + c, c + a) + (c, a, b) = (a + b + c)(1, 1, 1)$ .
  - (c)  $2i + 3j = 3j + 2i$ .
  - (d)  $i + j + k$  is a unit vector.
  - (e)  $0 - (a, b) = (-a, b)$ .
  - (f)  $\vec{OP} - \vec{OQ} = \vec{PQ}$ .
  - (g) If  $u$  and  $v$  are vectors, then  $uv$  has not been defined so far.
  - (h)  $\alpha 0 = 0$ .
  - (i)  $0u = 0$ .
  - (j) The zeros on both sides of (e) in § 2.2.7 are different.
- (k) Let  $ABC$  be a triangle. Then  $\vec{AB} + \vec{BC} + \vec{CA} = 0$ .

## 2.3 DOT PRODUCT OF VECTORS

We start this article with the definition of the angle between two vectors.

**2.3.1 Definition** Let  $u = \vec{OP}$  and  $v = \vec{OQ}$  be two nonzero vectors. Then the angle between  $u$  and  $v$ ,  $u \neq \lambda v$ , is defined as the angle of positive measure  $\theta$  between  $\vec{OP}$  and  $\vec{OQ}$  interior to the triangle  $POQ$  (Figure 2.17). If  $u = \lambda v$ , then the angle  $\theta$  is defined as  $\theta = 0$ , if  $\lambda > 0$ , and  $\theta = \pi$ , if  $\lambda < 0$ .

It follows from Definition 2.3.1 that  $0 \leq \theta \leq \pi$ . We shall now prove the following theorem, which gives the angle between two vectors.

**2.3.2 Theorem** Let  $u = (a_1, a_2, a_3) = \vec{OP}$  and  $v = (b_1, b_2, b_3) = \vec{OQ}$  be two nonzero vectors in  $V_3$ . Let  $\theta$  be the angle between  $u$  and  $v$ . Then

$$|\vec{OP}| |\vec{OQ}| \cos \theta = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (1)$$

*Proof :* If  $0 \leq \theta < \pi$ , consider the triangle  $POQ$  (see Figure 2.17).

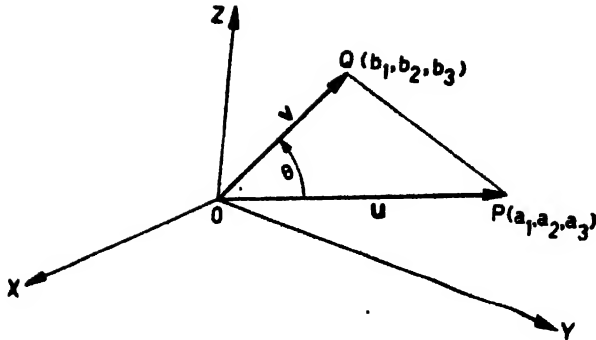


FIGURE 2.17

We have  $\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$ . From geometry, we have

$$\begin{aligned}
 |\overrightarrow{OP}|^2 + |\overrightarrow{OQ}|^2 - 2|\overrightarrow{OP}||\overrightarrow{OQ}|\cos\theta &= |\overrightarrow{QP}|^2 \\
 \text{or } 2|\overrightarrow{OP}||\overrightarrow{OQ}|\cos\theta &= |\overrightarrow{OP}|^2 + |\overrightarrow{OQ}|^2 - |\overrightarrow{QP}|^2 \\
 &= |(a_1, a_2, a_3)|^2 + |(b_1, b_2, b_3)|^2 \\
 &\quad - |(a_1 - b_1, a_2 - b_2, a_3 - b_3)|^2 \\
 &= (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) \\
 &\quad - (a_1 - b_1)^2 - (a_2 - b_2)^2 - (a_3 - b_3)^2 \\
 &= 2a_1b_1 + 2a_2b_2 + 2a_3b_3.
 \end{aligned}$$

Therefore,  $|\overrightarrow{OP}||\overrightarrow{OQ}|\cos\theta = a_1b_1 + a_2b_2 + a_3b_3$ .

Now let  $\theta = 0$  or  $\theta = \pi$ . In these cases we have  $\overrightarrow{OP} = \beta\overrightarrow{OQ}$ , where  $\beta > 0$ , if  $\theta = 0$ , and  $\beta < 0$ , if  $\theta = \pi$ , i.e.  $\beta = |\beta|\cos\theta$ . Thus,

$$(a_1, a_2, a_3) = \beta(b_1, b_2, b_3), \text{ i.e. } a_i = \beta b_i, i = 1, 2, 3.$$

$$\begin{aligned}
 \text{Hence } |\overrightarrow{OP}||\overrightarrow{OQ}|\cos\theta &= |\beta||\overrightarrow{OQ}|^2\cos\theta \\
 &= |\beta|(\cos\theta)(b_1^2 + b_2^2 + b_3^2) \\
 &= \beta(b_1^2 + b_2^2 + b_3^2) \\
 &= a_1b_1 + a_2b_2 + a_3b_3,
 \end{aligned}$$

because  $a_i = \beta b_i, i = 1, 2, 3$ .

A similar result can be obtained for plane vectors. We have only to suppress the third coordinates  $a_3$  and  $b_3$ . In this case, if  $u = (a_1, a_2) = \overrightarrow{OP}$  and  $v = (b_1, b_2) = \overrightarrow{OQ}$ , we have

$$|\overrightarrow{OP}||\overrightarrow{OQ}|\cos\theta = a_1b_1 + a_2b_2, \quad (2)$$

where  $\theta$  is the angle between the vectors  $u$  and  $v$ .

**Example 2.14** Let  $u = (3, 2)$  and  $v = (2, 5)$ . Then the angle  $\theta$  between the vectors  $u$  and  $v$  is given by

$$|(3, 2)| |(2, 5)| \cos \theta = 6 + 10.$$

So 
$$\cos \theta = \frac{16}{\sqrt{13}\sqrt{29}} = \frac{16}{\sqrt{377}}.$$

**Example 2.15** Let  $u = (1, -1, 3)$  and  $v = (3, 1, 1)$ . Then the angle  $\theta$  between the vectors  $u$  and  $v$  is given by  $|(1, -1, 3)| |(3, 1, 1)| \cos \theta = 3 - 1 + 3$ . So  $\cos \theta = 5/11$ .

Now we shall define the dot product of two vectors. Actually, there are two different ways of multiplying two vectors. In one the product is a real number, whereas in the other the product is a vector. In this article we shall consider the first kind of product. The second kind will be dealt with in Chapter 6.

**2.3.3 Definition** Let  $u = \overrightarrow{OP}$  and  $v = \overrightarrow{OQ}$  be two nonzero vectors and  $\theta$  be the angle between them. Then the *dot product* (also called *scalar product* or *inner product*), written as  $u \cdot v$ , is the real number defined by

$$u \cdot v = |\overrightarrow{OP}| |\overrightarrow{OQ}| \cos \theta = |u| |v| \cos \theta. \quad (3)$$

If either  $u$  or  $v$  is the zero vector, then  $u \cdot v = 0$ .

This definition is based on the geometric concept of the angle between two vectors. However, Theorem 2.3.2 gives an algebraic expression for the dot product of  $u$  and  $v$  in terms of their components. Hence, Definition 2.3.3 may be equivalently written as follows.

**2.3.4 Definition** Let  $u = (a_1, a_2)$  and  $v = (b_1, b_2)$  be two vectors of  $V_2$ . Then the dot product  $u \cdot v$  is defined by

$$u \cdot v = a_1 b_1 + a_2 b_2. \quad (4)$$

If  $u = (a_1, a_2, a_3)$  and  $v = (b_1, b_2, b_3)$  are vectors of  $V_3$ , then  $u \cdot v$  is defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + a_3 b_3. \quad (5)$$

Note that the formation of the dot product is *not* a binary operation. (Why?) However, it satisfies the following properties.

### 2.3.5 Properties of the Dot Product

Let  $u, v, w$  be vectors in  $V_2$  (or in  $V_3$ ) and  $\alpha, \beta$  be real numbers. Then

(a)  $u \cdot u = |u|^2$ ; hence  $u \cdot u \geq 0$ . (6)

(b)  $u \cdot u = 0$  iff  $u = 0$ . (7)

(c)  $u \cdot v = v \cdot u$ . (8)

(d)  $u \cdot (v + w) = u \cdot v + u \cdot w$ . (9)

(e)  $(\alpha u) \cdot v = \alpha (u \cdot v) = u \cdot (\alpha v)$ . (10)

**Proof :** We shall prove these properties for vectors in  $V_3$ . Analogously, by suppressing the third coordinate, we can prove these properties for vectors in  $V_2$ . Let  $u = (a_1, a_2, a_3)$ ,  $v = (b_1, b_2, b_3)$ ,  $w = (c_1, c_2, c_3)$ , and  $\alpha$  be a real number.

$$(a) \quad u \cdot u = a_1^2 + a_2^2 + a_3^2 = |u|^2.$$

$$(b) \quad \text{If } u = (0, 0, 0), \text{ then } u \cdot u = 0. \text{ Conversely, if } u \cdot u = a_1^2 + a_2^2 + a_3^2 = 0, \text{ then } a_1 = a_2 = a_3 = 0, \text{ because } a_1^2, a_2^2, \text{ and } a_3^2 \text{ are all nonnegative. Hence, } u = (0, 0, 0) = 0.$$

$$(c) \quad u \cdot v = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3 = v \cdot u$$

$$(d) \quad \begin{aligned} u \cdot (v + w) &= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3) \\ &= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3) \\ &= u \cdot v + u \cdot w. \end{aligned}$$

$$(e) \quad \begin{aligned} (\alpha u) \cdot v &= (\alpha a_1)b_1 + (\alpha a_2)b_2 + (\alpha a_3)b_3 \\ &= \alpha(a_1b_1) + \alpha(a_2b_2) + \alpha(a_3b_3) \\ &= \alpha(a_1b_1 + a_2b_2 + a_3b_3) = \alpha(u \cdot v). \end{aligned}$$

Similarly,  $u \cdot (\alpha v) = \alpha(u \cdot v)$ . ■

Another important property satisfied by the dot product is the following.

### 2.3.6 Schwarz Inequality

For any two vectors  $u$  and  $v$  of  $V_3$  (or  $V_2$ ), we have

$$|u \cdot v| \leq |u| |v|,$$

i.e.

$$(u \cdot v)^2 \leq (u \cdot u)(v \cdot v). \quad (11)$$

**Proof :** If either  $u$  or  $v$  is the zero vector, there is nothing to prove. So we assume that  $u$  and  $v$  are nonzero vectors. Let  $\theta$  be the angle between  $u$  and  $v$ . Then, by Definition 2.3.3, we have

$$\cos \theta = \frac{u \cdot v}{|u| |v|}$$

But  $|\cos \theta| \leq 1$ . Therefore,

$$|u \cdot v| \leq |u| |v|,$$

$$\text{i.e. } (u \cdot v)^2 \leq |u|^2 |v|^2 = (u \cdot u)(v \cdot v). \quad \blacksquare$$

**Example 2.16** The dot product of the vectors  $(2, 1)$  and  $(-1, 6)$  is  $(2, 1) \cdot (-1, 6) = -2 + 6 = 4$ .

**Example 2.17** The dot product of the vectors  $(1, -1, 3)$  and  $(0, 1, -3)$  is  $0 + (-1) + (-9) = -10$ .

**Example 2.18** For any real numbers  $a_1, a_2, a_3, b_1, b_2, b_3$ , we have  $(a_1b_1 + a_2b_2 + a_3b_3)^2 \leq (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$ .

(Hint : Apply Schwarz inequality to the vectors  $u = (a_1, a_2, a_3)$  and  $v = (b_1, b_2, b_3)$ .)

Applying the definition of dot product to the vectors  $i, j$ , and  $k$ , we have

$$i \cdot i = j \cdot j = k \cdot k = 1 \quad (\text{by 2.3.5 (i)})$$

and  $i \cdot j = j \cdot k = k \cdot i = 0$ .

We know that  $0 \cdot u = 0 = u \cdot 0$  for any vector  $u$ . The converse is *not* true, i.e.  $u \cdot v = 0$  need not imply that either of the vectors  $u$  or  $v$  is a zero vector. For,  $i \cdot j = 0$  but  $i$  and  $j$  are both nonzero vectors.

In general, if  $u$  and  $v$  are nonzero vectors and if  $u \cdot v = 0$ , then in view of Definition 2.3.3,  $\cos \theta = 0$ , where  $\theta$  is the angle between  $u$  and  $v$ . In other words,  $u$  and  $v$  are perpendicular to each other. Thus,  $u \cdot v = 0$  implies that either  $u = 0$  or  $v = 0$  or  $u$  and  $v$  are perpendicular to each other.

**2.3.7 Definition** Two vectors  $u$  and  $v$  of  $V_2$  (or  $V_3$ ) are said to be *orthogonal* if  $u \cdot v = 0$ .

Note that  $0$  is orthogonal to every vector  $u$  and  $i, j, k$  are pairwise orthogonal.

**Example 2.19** Let  $u = (1, 2, 3)$  and  $v = (2, -7, 4)$ . Then  $u \cdot v = 2 - 14 + 12 = 0$ . So  $u$  and  $v$  are orthogonal.

**Example 2.20** Let  $u = (1, 2, 3)$  and  $v = (2, -5, 4)$ . Then  $u \cdot v = 2 - 10 + 12 \neq 0$ . So  $u$  and  $v$  are not orthogonal.

Every vector  $u = (a_1, a_2)$  in a plane can be expressed as the sum of two orthogonal vectors  $a_1 i$  and  $a_2 j$ .  $a_1 i$  and  $a_2 j$  are called the resolved parts of  $u$  respectively along and perpendicular to the  $x$ -axis. In general, let  $u$  and  $v$  be two vectors. If we express  $u$  as a sum of two vectors  $w_1$  and  $w_2$  such that  $w_1$  is parallel to  $v$ , and  $w_2$  is orthogonal to  $v$ , then  $w_1$  and  $w_2$  are called the resolved parts of  $u$  respectively along and perpendicular to  $v$ . In such a case, we have

$$u = w_1 + w_2,$$

where  $w_1 = \lambda v$  for some real number  $\lambda$  and  $w_2 \cdot v = 0$ .

Hence,

$$\begin{aligned} u \cdot v &= (w_1 + w_2) \cdot v = w_1 \cdot v + w_2 \cdot v \\ &= (\lambda v) \cdot v = \lambda(v \cdot v). \end{aligned}$$

Thus,

$$\lambda = \frac{u \cdot v}{v \cdot v}, \text{ if } v \cdot v \neq 0.$$

Therefore,

$$w_1 = \frac{u \cdot v}{v \cdot v} v \text{ and } w_2 = u - w_1 = u - \frac{u \cdot v}{v \cdot v} v \quad (\text{Figure 2.18}).$$

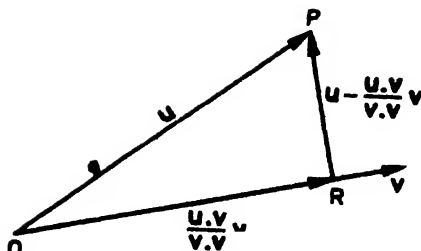


FIGURE 2.18

We have thus proved the following theorem.

**2.3.8 Theorem** *Let  $v$  be a given nonzero vector. Then any nonzero vector  $u$  can be expressed as the sum of a vector parallel to  $v$  and a vector orthogonal to  $v$ .*

In the foregoing discussion the vector  $w_1 = \frac{u \cdot v}{v \cdot v} v$  is called the *vector projection of  $u$  along  $v$* .

The *scalar projection of  $u$  on  $v$*  is  $|u| \cos \theta$ , where  $\theta$  is the angle between  $u$  and  $v$ . Obviously, the scalar projection of  $u$  on  $v$  is  $\frac{u \cdot v}{|v|}$  and the vector projection of  $u$  along  $v$  is

$$\begin{aligned} \frac{u \cdot v}{v \cdot v} v &= \frac{u \cdot v}{|v|} \frac{v}{|v|} \\ &= (\text{scalar projection of } u \text{ on } v) \text{ times the unit vector in the direction of } v. \end{aligned}$$

Thus, the magnitude of the vector projection of  $u$  along  $v$  is the absolute value of the scalar projection of  $u$  on  $v$  and the direction is the same as that of  $v$ , if  $\theta < \pi/2$ , and opposite to that of  $v$ , if  $\theta > \pi/2$ . If  $\theta = \pi/2$ , the vector projection of  $u$  along  $v$  is  $0$ .

**Example 2.21** Let  $u = (1, 2, 3) = i + 2j + 3k$  and  $v = (-2, 3, 0) = -2i + 3j + 0k$ . Then

$$\text{scalar projection of } u \text{ on } v \text{ is } \frac{u \cdot v}{|v|} = \frac{-2 + 6}{\sqrt{13}} = \frac{4}{\sqrt{13}}$$

$$\begin{aligned} \text{vector projection of } u \text{ along } v \text{ is } \frac{u \cdot v}{v \cdot v} v &= \frac{4}{13} (-2, 3, 0) \\ &= -\frac{8}{13} i + \frac{12}{13} j + 0k. \end{aligned}$$

Also

$$\begin{aligned} u = (1, 2, 3) = i + 2j + 3k &= \left(-\frac{8}{13} i + \frac{12}{13} j + 0k\right) \\ &\quad + \left(\frac{21}{13} i + \frac{14}{13} j + 3k\right). \end{aligned}$$



This is the resolution of  $u$  along and perpendicular to  $v$ . The reader can check that  $(-\frac{8}{13}i + \frac{12}{13}j + 0k)$  is along  $v$  and  $(\frac{21}{13}i + \frac{14}{13}j + 3k)$  is perpendicular to  $v$ .

### Problem Set 2.3

Work out Problems 1 through 4 for each pair of vectors  $u$  and  $v$  given as follows.

- (a)  $u = (1, 1)$ ,  $v = (-1, 3)$
- (b)  $u = (-2, 5)$ ,  $v = (1, -6)$
- (c)  $u = i + 2j$ ,  $v = 2i - 3j$
- (d)  $u = -3i + 5j$ ,  $v = 6i + 4j$
- (e)  $u = (1, 3, 5)$ ,  $v = (-3, 1, 1)$
- (f)  $u = (2, -1, 4)$ ,  $v = (1, 2, -1)$
- (g)  $u = -i + j + k$ ,  $v = 3i - 2j + k$
- (h)  $u = 2i - 3j + k$ ,  $v = i + j + k$ .

1. Find  $u \cdot v$ .
2. Find the cosine of the radian measure of the angle between  $u$  and  $v$ .
3. Find the scalar projection of  $u$  on  $v$ .
4. Find the vector projection of  $v$  along  $u$ .
5. Find the real number  $\alpha$  such that the vectors  $u$  and  $v$  given as follow are orthogonal.

- (a)  $u = (2, \alpha, 1)$ ,  $v = (4, -2, -2)$
- (b)  $u = (1, -2, 1)$ ,  $v = (2, 1, \alpha)$
- (c)  $u = (0, 3, -2)$ ,  $v = (\alpha, 4, 6)$
- (d)  $u = (\alpha, -3, 1)$ ,  $v = (\alpha, \alpha, 2)$ .

6. For any vector  $u$ , prove that

$$u = (u \cdot i)i + (u \cdot j)j + (u \cdot k)k.$$

7. Let  $u$  and  $v$  be two vectors of  $V_3$  (or  $V_2$ ). Prove that

$$(a) \quad |u + v| \leq |u| + |v|$$

$$(b) \quad |u \cdot v| = |u| |v| \text{ iff } u \text{ is parallel to } v.$$

8. Let  $u$  be orthogonal to both  $v$  and  $w$ . Then prove that  $u$  is orthogonal to each vector of the set  $\{\alpha v + \beta w \mid \alpha, \beta \text{ any real numbers}\}$ .
9. True or false ?

- (a)  $u \cdot v = 0$  implies either  $u = 0$  or  $v = 0$ .
- (b) If  $u \cdot v = 0$  for all  $v \in V_3$ , then  $u = 0$ .
- (c)  $u \cdot (v \cdot w)$  is meaningless.

- (d)  $(u \cdot v)w$  is a vector parallel to  $w$ .  
 (e) In space if  $u \cdot j = 0$  and  $u \cdot k = 0$ , then  $u = i$ .  
 (f)  $(1, -1) \cdot (-1, 1, 0) = 0$ .  
 (g)  $|u| = 0$  iff  $u = 0$ .

## 2.4 APPLICATIONS TO GEOMETRY

The main applications that we consider in this article are equations to lines and planes and certain associated problems. First, we shall work out a few examples to illustrate the power of vector methods.

**Example 2.22** In trigonometry, the formula

$$\cos(\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi$$

is fundamental. We shall prove this formula by using vectors.

**Proof:** The unit vectors  $\vec{OQ}$  and  $\vec{OP}$  (see Figure 2.19) at angles  $\theta$  and  $\phi$

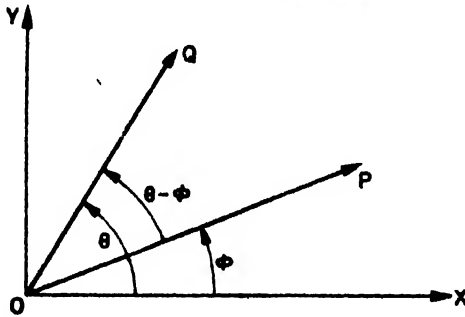


FIGURE 2.19

with the positive direction of the  $x$ -axis are respectively

$$\cos \theta i + \sin \theta j \quad \text{and} \quad \cos \phi i + \sin \phi j.$$

The angle  $\theta - \phi$  between the vectors  $\vec{OQ}$  and  $\vec{OP}$  is given by

$$\cos(\theta - \phi) = \frac{\vec{OP} \cdot \vec{OQ}}{|\vec{OP}| |\vec{OQ}|} = \cos \theta \cos \phi + \sin \theta \sin \phi.$$

**Example 2.23** Prove, by using vector algebra, that the three altitudes of a triangle are concurrent.

**Proof:** Let  $AD$  and  $BE$  be the two altitudes of a triangle  $ABC$ , meeting at the point  $O$  (see Figure 2.20). It is sufficient to prove that  $CO$  is perpendicular to  $AB$ .

Clearly,

$$\vec{AO} \cdot \vec{BC} = 0,$$

and

$$\vec{BO} \cdot \vec{CA} = 0.$$

(1)

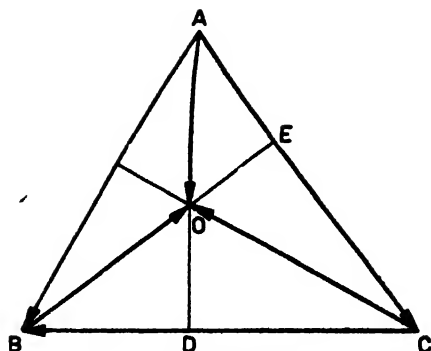


FIGURE 2.20

Note that we are working with vectors that do not have the same initial point. Now

$$\begin{aligned}
 \vec{CO} \cdot \vec{AB} &= \vec{CO} \cdot (\vec{AC} + \vec{CB}) \\
 &= \vec{CO} \cdot \vec{AC} + \vec{CO} \cdot \vec{CB} \\
 &= (\vec{CB} + \vec{BO}) \cdot \vec{AC} + (\vec{CA} + \vec{AO}) \cdot \vec{CB} \\
 &= \vec{CB} \cdot \vec{AC} + \vec{BO} \cdot \vec{AC} + \vec{CA} \cdot \vec{CB} + \vec{AO} \cdot \vec{CB} \\
 &= \vec{CB} \cdot \vec{AC} + \vec{CA} \cdot \vec{CB} \quad (\text{by (1)}) \\
 &= \vec{CB} \cdot \vec{AC} - \vec{AC} \cdot \vec{CB} \quad (\text{because } \vec{CA} = -\vec{AC}) \\
 &= \vec{CB} \cdot \vec{AC} - \vec{CB} \cdot \vec{AC} = 0 \quad (\text{by 2.3.5 (c)}).
 \end{aligned}$$

We shall now take up the standard applications of vectors to equations of lines and planes in analytic geometry. First, we note that if  $P(x_1, y_1)$  and  $Q(x_2, y_2)$  are two points in a plane, then the vector  $\vec{PQ}$  is given by

$$\vec{PQ} = \vec{OQ} - \vec{OP} = (x_2 - x_1, y_2 - y_1) = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}.$$

Similarly, if  $P(x_1, y_1, z_1)$  and  $Q(x_2, y_2, z_2)$  are two points in space, then the vector  $\vec{PQ}$  is  $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$ .

#### 2.4.1 The Equation of a Straight Line

Consider a straight line  $L$  parallel to a vector  $\mathbf{u}$  and passing through a point  $P$  (Figure 2.21).

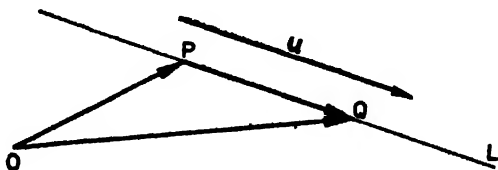


FIGURE 2.21

Let  $Q$  be a point on  $L$ . The vector  $r = \overrightarrow{OQ}$  ( $O$  is the origin) is called the position vector of point  $Q$ . Let  $v$  be the position vector of point  $P$ . Then

$$\begin{aligned} r &= \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} \\ &= v + \text{a vector parallel to } u \\ &= v + tu, \end{aligned} \quad (2)$$

where  $t$  is a real number. This equation is satisfied by all points  $Q$  on the line and by no points off the line. (Why?) It is called the vector equation of a straight line through a given point  $P$  and parallel to a given vector  $u$ . The following special cases are worth noting.

### 2.4.2 Line in a Plane

Let  $P$  be the point  $(x_1, y_1)$  and  $u$  be the vector  $(a, b)$ . Then  $v = \overrightarrow{OP} = (x_1, y_1)$  and  $r = OQ = (x, y)$ . Then the equation of the line takes the form

$$(x, y) = (x_1, y_1) + t(a, b) = (x_1 + ta, y_1 + tb).$$

This equation can be written as a pair of equations :

$$x = x_1 + ta, \quad y = y_1 + tb,$$

i.e.

$$(x - x_1) = ta, \quad (y - y_1) = tb.$$

These are called the parametric equations of the straight line. Eliminating  $t$ , we get the cartesian equation

$$a(y - y_1) = b(x - x_1). \quad (3)$$

If  $a = 0$  or  $b = 0$ , vector  $u$  becomes a vector along one of the coordinate axes and the line has the equation  $x = x_1$  or  $y = y_1$ .

If  $a \neq 0$  and  $b \neq 0$ , Equation (3) may be written in the symmetric form as

$$\frac{x - x_1}{a} = \frac{y - y_1}{b}. \quad (4)$$

If  $u$  is a unit vector at an angle  $\theta$  ( $\neq \pi/2$ ), then  $u = (a, b) = (\cos \theta, \sin \theta)$ . Hence, Equation (3) may be written as

$$y - y_1 = \tan \theta (x - x_1)$$

or

$$y - y_1 = m(x - x_1), \quad (5)$$

where  $m$  is the slope of the straight line  $L$ . Thus (5) is the equation of the straight line through  $P(x_1, y_1)$  and having slope  $m$ .

Thus, given a vector, we have found the equation to a straight line parallel to the vector. The converse situation is contained in the following example.

**Example 2.24** Let  $lx + my + n = 0$  be the equation of the straight line  $L$  in a plane. Then the vector  $-mi + lj$  is parallel to  $L$  and the vector  $li + mj$  is perpendicular to  $L$ .

**Case 1**  $n \neq 0$ .

If either  $l = 0$  or  $m = 0$ , then the straight line  $L$  is parallel to one of the coordinate axes and the result is true.

Now suppose that  $l \neq 0$  and  $m \neq 0$ . In this case the two points  $P(-n/l, 0)$  and  $Q(0, -n/m)$  lie on  $L$ . The vector  $\overrightarrow{QP}$  is  $-\frac{n}{l}i + \frac{n}{m}j$ .

The vector  $\frac{lm}{n}\overrightarrow{QP}$  is parallel to the vector  $\overrightarrow{QP}$  and hence to the line  $L$ . This vector is  $-mi + lj$ .

The vector  $li + mj$  is perpendicular to the vector  $\overrightarrow{QP}$  and hence to the line  $L$ , because

$$(li + mj) \cdot \left(-\frac{n}{l}i + \frac{n}{m}j\right) = -n + n = 0.$$

**Case 2**  $n = 0$ .

Shift the line  $lx + my = 0$  parallel to itself and obtain  $lx + my + n' = 0$ ,  $n' \neq 0$ . Using Case 1, we get the desired result.

### 2.4.3 Line in Space

Let  $P$  be the point  $(x_1, y_1, z_1)$  and  $u$  be the vector  $(a, b, c)$ . Then

$$r = \overrightarrow{OP} = (x_1, y_1, z_1)$$

and  $(x, y, z) = (x_1, y_1, z_1) + t(a, b, c)$ .

$r = (x, y, z)$  is the position vector of any point  $Q$  on the line  $L$ . This vector equation may be written in the parametric form

$$x = x_1 + ta; \quad y = y_1 + tb; \quad z = z_1 + tc. \quad (6)$$

Eliminating  $t$ , we get the symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}, \quad (7)$$

provided  $a, b$ , and  $c$  are nonzero. Thus, we get another form of the equation of a straight line through  $P(x_1, y_1, z_1)$  and parallel to the vector  $u = ai + bj + ck$ .

If any two of the numbers  $a, b, c$  are zero, say  $a = b = 0$ , then the vector  $u = (0, 0, c)$  becomes a vector along the  $z$ -axis. The equation of the straight line, from Equation (6), is

$$x = x_1, \quad y = y_1. \quad (8)$$

Note that these two together represent the straight line.

If just one of the numbers  $a, b, c$  is zero, say  $a = 0$ , then  $\mathbf{u}$  is the vector  $b\mathbf{j} + c\mathbf{k}$ . The equation of the straight line, from Equation (6), is

$$x = x_1, \quad \frac{y - y_1}{b} = \frac{z - z_1}{c}. \quad (9)$$

These two together represent the straight line.

#### 2.4.4 Direction Cosines

Let  $\alpha, \beta, \gamma$  be the inclinations of the vector  $\mathbf{u} = \overrightarrow{OA} = (a, b, c)$  to the positive directions of the coordinate axes  $OX, OY, OZ$ , respectively, as in Figure 2.22. Then  $\cos \alpha, \cos \beta, \cos \gamma$  are called the direction cosines of

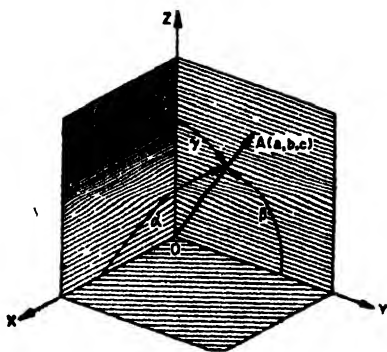


FIGURE 2.22

the vector  $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ . These are also called the direction cosines of a line parallel to  $\mathbf{u}$  and in the same direction as  $\mathbf{u}$ . The ordered set  $\{\cos \alpha, \cos \beta, \cos \gamma\}$  is called the set of direction cosines of the line represented by vector  $\mathbf{u}$ . Clearly,

$$\cos \alpha = \frac{\mathbf{u} \cdot \mathbf{i}}{|\mathbf{u}| |\mathbf{i}|} = \frac{a}{\sqrt{a^2 + b^2 + c^2}} \quad (10)$$

$$\cos \beta = \frac{\mathbf{u} \cdot \mathbf{j}}{|\mathbf{u}| |\mathbf{j}|} = \frac{b}{\sqrt{a^2 + b^2 + c^2}} \quad (11)$$

$$\cos \gamma = \frac{\mathbf{u} \cdot \mathbf{k}}{|\mathbf{u}| |\mathbf{k}|} = \frac{c}{\sqrt{a^2 + b^2 + c^2}}. \quad (12)$$

It follows that  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ . For any real number  $k$ , the ordered set  $\{k \cos \alpha, k \cos \beta, k \cos \gamma\}$  is called the set of direction ratios of a line parallel to  $\mathbf{u}$ . Clearly,  $a, b, c$  are the direction ratios of the line parallel to the vector  $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  and hence of the line

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

### 2.4.5 The Equation of a Plane

Consider a plane passing through a point  $Q(x_1, y_1, z_1)$  and perpendicular to a vector  $\mathbf{u} = \overrightarrow{OA} = (a, b, c)$  (see Figure 2.23). Let  $P(x, y, z)$

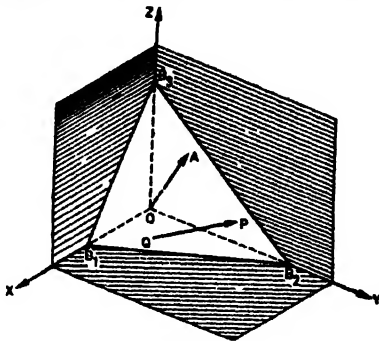


FIGURE 2.23

be a point in the plane. Then the vector  $\overrightarrow{QP} = (x - x_1)\mathbf{i} + (y - y_1)\mathbf{j} + (z - z_1)\mathbf{k}$  lies in the plane and is perpendicular to the vector  $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ . Therefore,

$$\mathbf{u} \cdot \overrightarrow{QP} = 0,$$

which gives

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0. \quad (13)$$

This is satisfied by all points  $P$  on the plane and by no points off the plane. Equation (13) is called the equation of the plane through  $Q(x_1, y_1, z_1)$  and perpendicular to the vector  $\mathbf{u} = (a, b, c)$ .

Equation (13) of the plane can further be simplified to the form

$$ax + by + cz + d = 0,$$

where  $d = -ax_1 - by_1 - cz_1$ . Thus, the equation of a plane in space is a linear equation in  $x, y$ , and  $z$ . Conversely, consider a linear equation

$$Ax + By + Cz + D = 0, \quad (14)$$

where  $A, B$ , and  $C$  are not all zero. Let  $A \neq 0$ . Then we can write Equation (14) as

$$A\left(x + \frac{D}{A}\right) + B(y - 0) + C(z - 0) = 0,$$

which is the equation of a plane through the point  $(-D/A, 0, 0)$  and perpendicular to the vector  $A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ .

Two planes, if not parallel, intersect in a straight line. Thus, a straight line may also be considered as the intersection of two planes, and hence

its equation may also be given as a pair of equations of planes, e.g.

$$a_1x + b_1y + c_1z + d_1 = 0$$

and

$$a_2x + b_2y + c_2z + d_2 = 0$$

represent a line. From these equations, we can get the equation of the line in the symmetric form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

We illustrate this by the following example.

**Example 2.25** Find the symmetric form of the equation of the line of intersection of planes

$$x + 3y - z + 5 = 0 \quad (15)$$

and

$$5x - 2y + 4z - 8 = 0. \quad (16)$$

Eliminating  $z$  from the two equations, we get

$$9x + 10y + 12 = 0$$

or

$$9x = -10y - 12.$$

Eliminating  $x$  from Equations (15) and (16), we get

$$y - \frac{9z}{17} + \frac{33}{17} = 0$$

or

$$-10y - 12 = -\frac{90z}{17} + \frac{126}{17}$$

So, we get

$$9x = -10y - 12 = -\frac{90z - 126}{17}$$

or

$$\frac{x - 0}{1/9} = \frac{y + 6/5}{-1/10} = \frac{z - 7/5}{-17/90}$$

or

$$\frac{x - 0}{-10} = \frac{y + 6/5}{9} = \frac{z - 7/5}{17}$$

### Problem Set 2.4

1. Prove by vectors that the diagonals of a rhombus meet at right angles.
2. Prove by vectors that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.
3. Find the direction cosines of the line passing through the points  $A(1, 2, 1)$  and  $B(3, -1, -1)$ .
4. Find the vector equation of the line passing through two given points.
5. Prove that the points  $A, B, C$  are collinear iff  $\vec{OC} = \alpha\vec{OA} + \beta\vec{OB}$ , where  $\alpha + \beta = 1$ .



6. Prove that the perpendicular distance from the point  $P$  to the line

$$r = v + tu \text{ is } \left| v + \frac{(\vec{OP} - v) \cdot u}{u \cdot u} u - \vec{OP} \right|.$$

7. Find the angles between the lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}, \quad \frac{x-a'}{l'} = \frac{y-b'}{m'} = \frac{z-c'}{n'}.$$

8. Find the angles that the vector  $u = 2i - 2j + k$  makes with the coordinate axes.
9. Find the equation of the plane perpendicular to the vector  $u = 2i + 3j + 6k$  and passing through the terminal point of the vector  $i + 5j + 3k$ .
10. Determine a vector perpendicular to the plane containing the vectors  $u = 2i - 6j - 3k$  and  $v = 4i + 3j - k$ .
11. Find the angle between the planes
- (a)  $2x - y + 2z = 1$  and  $x - y = 2$
- (b)  $3x + 4y - 5z = 9$  and  $2x + 6y + 6z = 7$ .
12. Prove that the distance of the point  $P_0(x_0, y_0, z_0)$  from the plane  $ax + by + cz + d = 0$  is

$$\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$

13. Prove that the equation of the line of intersection of the planes  $4x + 4y - 5z = 12$  and  $8x + 12y - 13z = 32$  can be written in the form

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}.$$

14. Prove that the equation of the sphere of radius  $r$  with centre  $P(x_1, y_1, z_1)$  is  $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = r^2$ .

# Vector Spaces

In Chapter 2 we saw that the set of all plane (space) vectors forms a commutative group relative to addition and, further, relative to scalar multiplication it satisfies certain properties (Properties 2.2.7). All these properties of vectors are so fundamental in mathematics that whenever any system satisfies them we give it a special name, namely, vector space. The precise definition of a vector space follows.

## 3.1 VECTOR SPACES

**3.1.1 Definition** A nonempty set  $V$  is called a *real vector space* (or a *real linear space* or a *real linear vector space*) if the following axioms are satisfied :

VS1 There is a binary operation '+' defined on  $V$ , called 'addition'.

VS2 There is a scalar multiplication defined on  $V$ . (This means, to every real number  $\alpha$  and every element  $u$  of  $V$ , we can associate a unique element of  $V$  and denote it by  $\alpha u$ )

VS3 Addition and scalar multiplication satisfy the following :

(a)  $V$  is a commutative group for addition (i.e. G1, G2, G3, and G4 hold for addition in  $V$ ).

(b)  $\alpha(u + v) = \alpha u + \alpha v$  and  $(\alpha + \beta)u = \alpha u + \beta u$  for all real numbers  $\alpha, \beta$  and all  $u, v \in V$ .

(c)  $\alpha(\beta u) = (\alpha\beta)u = \beta(\alpha u)$  for all real numbers  $\alpha, \beta$  and all  $u \in V$ .

(d)  $1u = u$  for all  $u \in V$ .

**3.1.2 Remark** A *complex vector space* is defined analogously by using complex numbers instead of real numbers in Definition 3.1.1.

**3.1.3 Remark** The real or complex numbers used for scalar multiplication in the definition of a vector space are called scalars. Throughout we shall deal with only real vector spaces and complex vector spaces.

In Chapter 2 we saw that  $V_2$  and  $V_3$  have addition and scalar multiplication defined on them and VS3(a) holds for addition in both (Theorem 2.2.5); and that (b), (c), and (d) of VS3 hold for scalar multiplication in  $V_2$  and  $V_3$  (Properties 2.2.7). Since these properties of plane vectors and space vectors have been abstracted to provide the general definition of a vector space, we shall hereafter use the word 'vector' to mean 'an element of a vector space'. In other words, elements of a vector space shall be called vectors. Plane vectors and space vectors are only specific cases.

**3.1.4 Remark** Note that here we have not used  $u, v, w$  for denoting vectors as we did in Chapter 2. We shall consistently use the letters  $u, v, w, \dots$  for vectors, and the Greek letters  $\alpha, \beta, \gamma, \dots$  for scalars. However, we use the boldface  $\mathbf{0}$  to denote the zero vector of the space to distinguish it from the scalar 'zero'.

Clearly,  $V_2$  and  $V_3$  are vector spaces. There are many other examples of vector spaces. We shall note a few of them here. In fact, the reader should be able to visualise several vector spaces before we go deeper into the subject. Visualisation of a vector space involves five steps :

- (i) Consider a nonempty set  $V$ .
- (ii) Define a binary operation on  $V$  and call it 'addition'.
- (iii) Define scalar multiplication on  $V$  as described in VS2.
- (iv) Define equality in  $V$ .
- (v) Check whether (a) to (d) of VS3 are satisfied relative to the addition and scalar multiplication thus defined.

**Example 3.1** Let  $V_n$  be the set of all ordered  $n$ -tuples of real numbers. Thus, an element of  $V_n$  would be of the form  $(x_1, x_2, \dots, x_n)$ , where the  $x_i$ 's are real numbers. Define addition, scalar multiplication, and equality in  $V_n$  as follows : If  $u = (x_1, x_2, \dots, x_n)$  and  $v = (y_1, y_2, \dots, y_n)$  are two elements of  $V_n$ , then :

**Addition**

$$u + v = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n). \quad (1)$$

(Note that  $u + v \in V_n$ , because the right-hand side of Equation (1) is an ordered  $n$ -tuple of real numbers. This is called coordinatewise addition.)

**Scalar multiplication** For a real scalar  $\alpha$ ,  $\alpha u$  is defined as the ordered  $n$ -tuple  $(\alpha x_1, \alpha x_2, \dots, \alpha x_n)$ , i.e.

$$\alpha u = (\alpha x_1, \alpha x_2, \dots, \alpha x_n), \quad (2)$$

which is again in  $V_n$ . (This is called coordinatewise scalar multiplication.)

**Equality**  $u = (x_1, x_2, \dots, x_n)$  and  $v = (y_1, y_2, \dots, y_n)$  are defined as equal if  $x_i = y_i, i = 1, 2, \dots, n$

Since addition, scalar multiplication, and equality in  $V_n$  have been

defined, what remains is to check whether (a) to (d) of VS3 are satisfied. We shall leave most of this checking to the reader. However, in order to give him an idea of what such checking involves, we shall now go through some of the details.

### *Commutativity of addition*

Let  $u = (x_1, x_2, \dots, x_n)$  and  $v = (y_1, y_2, \dots, y_n)$  be two members of  $V_n$ .

Then  $u + v = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$

and  $v + u = (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$ .

The real numbers  $x_k + y_k$  and  $y_k + x_k$  are equal, since the commutative law of addition holds for real numbers. As this argument is true for every  $k$ , it follows from the definition of equality that  $u + v = v + u$ .

The alert reader would have now noted that, fundamentally, it is the commutative law of addition among real numbers that gives the commutative law of addition in  $V_n$ . If this fact is remembered, we can check that  $u + v = v + u$  even mentally.

### *Existence of additive identity*

We have to determine an element  $0$ , which has the property  $0 + u = u$ . In this case we find that  $0 = (0, 0, \dots, 0)$ , because  $(0, 0, \dots, 0) + (x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)$  for all  $(x_1, x_2, \dots, x_n) \in V_n$ .

### *Existence of additive inverses*

Here we have to determine an element  $-u$  for the vector  $u \in V_n$  such that  $-u + u = 0 = (0, 0, \dots, 0)$ . If  $u = (x_1, x_2, \dots, x_n)$ , then we can easily see that  $-u = (-x_1, -x_2, \dots, -x_n)$ .

We assume that the reader can check the remaining properties. Once (a) to (d) of VS3 are checked, it follows that  $V_n$ , for the addition and scalar multiplication defined in (1) and (2), is a vector space.

It may be noted that, by Theorem 1.6.3,  $0 = (0, 0, \dots, 0)$  is the unique zero of the space and  $-u = (-x_1, -x_2, \dots, -x_n)$  is the unique negative of  $u = (x_1, x_2, \dots, x_n)$ .

Thus,  $V_n$  is a real vector space, since we have used only real scalars. Can we use complex scalars in this case? No. For, suppose  $\alpha$  is a complex number. Then  $\alpha u = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$  is not in  $V_n$ , because the numbers  $\alpha x_k$  are complex and  $V_n$  contains only  $n$ -tuples of real numbers; so scalar multiplication is not defined. See, however, Problem 2.

The special cases  $n = 2$  and  $n = 3$  of Example 3.1 give us the vector spaces  $V_2$  and  $V_3$ . The special case  $n = 1$  gives the space  $V_1$ , which is nothing but the space of real numbers, where addition is the ordinary addition of real numbers and scalar multiplication is the ordinary multiplication of real numbers.

**Example 3.2** Let  $\mathcal{F}(I)$  be the set of all real-valued functions defined on the interval  $I$ . With pointwise addition and scalar multiplication (cf § 1.7),  $\mathcal{F}(I)$  becomes a real vector space. The zero of this space is the function  $0$  given by  $0(x) = 0$  for all  $x \in I$ .

**Example 3.3** If, instead of the real-valued functions in Example 3.2, we use complex valued functions defined on  $I$  and pointwise addition and scalar multiplication, then we get a complex vector space (using complex scalars). We denote this complex vector space by  $\mathcal{F}_C(I)$ .

**Example 3.4** Let  $\mathcal{P}(I)$  denote the set of all polynomials  $p$  with real coefficients defined on the interval  $I$ . Recall (cf Example 1.39)  $p$  is a function whose value at  $x$  is

$$p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n \quad \text{for all } x \in I,$$

where  $\alpha_i$ 's are real numbers and  $n$  is a nonnegative integer. Using pointwise addition and scalar multiplication as for functions, we find that  $\mathcal{P}(I)$  is a real vector space. If we take complex coefficients for the polynomials and use complex scalars, then we get the complex vector space  $\mathcal{P}_C(I)$ . In both cases the vector  $0$  of the space is the zero polynomial given by  $0(x) = 0$  for all  $x \in I$ .

## NOTATIONS

$\mathcal{C}[a, b]$  = the set of all real-valued functions defined and continuous on the closed interval  $[a, b]$ .

$\mathcal{C}^{(1)}[a, b]$  = the set of all real-valued functions defined on  $[a, b]$  and whose first derivatives are continuous on  $[a, b]$ .

$\mathcal{C}^{(n)}[a, b]$  = the set of all real-valued functions defined on  $[a, b]$ , differentiable  $n$ -times and whose  $n$ -th derivatives are continuous on  $[a, b]$ . These functions are called  $n$ -times continuously differentiable functions.

**Example 3.5**  $\mathcal{C}[a, b]$ ,  $\mathcal{C}^{(1)}[a, b]$ ,  $\mathcal{C}^{(n)}[a, b]$  are real vector spaces under pointwise addition and scalar multiplication. We have to use the fact from calculus that the sum of two continuous (differentiable) functions is continuous (differentiable) and any scalar multiple of a continuous (differentiable) function is continuous (differentiable).

By changing the domain of definitions of continuity and differentiability to the open interval  $(a, b)$ , we get, similarly, the real vector spaces  $\mathcal{C}(a, b)$  and  $\mathcal{C}^{(n)}(a, b)$  for each positive integer  $n$ .

**3.1.5 Remark** By changing real-valued functions to complex-valued functions and using complex scalars, we get the complex vector spaces  $\mathcal{C}_C[a, b]$  and  $\mathcal{C}_C(a, b)$ .

**3.1.6 Remark** In each of these cases the reader should convince himself that the sets concerned are 'vector spaces' under the definitions

mentioned. He should check the axioms in each case until he reaches the stage where he can quickly say whether the axioms are satisfied or not. In any case no axiom should be taken for granted. In all cases the zero element of the space and the negative of any given element of the space should be clear in the reader's mind.

**Example 3.6** Let  $\mathcal{C}^{(\infty)}[a, b]$  stand for the set of all functions defined on  $[a, b]$  and having derivatives of all orders on  $[a, b]$ . This is a real vector space for the usual operations. It is called the space of infinitely differentiable functions on  $[a, b]$ .

We conclude this article with a theorem giving certain immediate consequences of the definition of a vector space. Recall that, in the definition of a vector space, we have property (d) of VS3, namely,  $1u = u$  for all  $u \in V$ . It should be noted that this apparently trivial axiom is crucial to the development of the theory of vector spaces.

**3.1.7 Theorem** In any vector space  $V$ ,

- (a)  $\alpha 0 = 0$  for every scalar  $\alpha$ .
- (b)  $0u = 0$  for every  $u \in V$ .
- (c)  $(-1)u = -u$  for every  $u \in V$ .

*Proof:* (a)  $\alpha 0 = \alpha(0 + 0)$  (by G2)  
 $= \alpha 0 + \alpha 0$  (by VS3(b)).

Adding  $-(\alpha 0)$  to both sides, we get

$$\begin{aligned} 0 &= -(\alpha 0) + (\alpha 0 + \alpha 0) \\ &= (-(\alpha 0) + (\alpha 0)) + \alpha 0 && \text{(by G1)} \\ &= 0 + \alpha 0 && \text{(by G3)} \\ &= \alpha 0 && \text{(by G2).} \end{aligned}$$

$$(b) \quad 0u = (0 + 0)u = 0u + 0u \quad \text{(by VS3(b)).}$$

Adding  $-(0u)$  to both sides, we get

$$\begin{aligned} 0 &= -(0u) + (0u + 0u) = (-(0u) + (0u)) + 0u && \text{(by G1)} \\ &= 0 + 0u && \text{(by G3)} \\ &= 0u && \text{(by G2).} \end{aligned}$$

$$\begin{aligned} (c) \quad (-1)u + u &= (-1)u + 1u && \text{(by VS3(d))} \\ &= (-1 + 1)u && \text{(by VS3(c))} \\ &= 0u = 0 && \text{(by (b)).} \end{aligned}$$

So by G3 and the uniqueness of the negative,  $(-1)u$  is the negative of  $u$ , i.e.  $(-1)u = -u$ . ■

It is convenient to write  $u - v$  for  $u + (-v)$ .

## Problem Set 3 1

1. Let  $u_1 = (1, 3, 2, 7)$ ,  $u_2 = (0, 2, -3, 5)$ ,  $u_3 = (-1, 3, 1, -4)$ , and  $u_4 = (-2, 16, -1, 5)$  be vectors of  $V_4$ . Then evaluate
- (a)  $u_1 + u_2$     (b)  $2u_1 + 3u_2 - 7u_4$     (c)  $u_1 + 2u_2 + 3u_3 - u_4$   
 (d)  $3u_2 + (u_4 - u_1)$     (e)  $(au_1 - bu_2) + au_3$ .

2. Consider the set  $V_n^C$  of all ordered  $n$ -tuples of complex numbers. By defining equality, addition, and scalar multiplication (with complex scalars) as in  $V_n^C$ , prove that  $V_n^C$  is a complex vector space. Is it a real vector space?

3. Let  $R^+$  be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

$$u + v = u \cdot v \quad \text{for all } u, v \in R^+ \\ \alpha u = u^\alpha \quad \text{for all } u \in R^+ \text{ and real scalar } \alpha.$$

Prove that  $R^+$  is a real vector space.

4. Which of the following subsets of  $V_4$  are vector spaces for coordinate-wise addition and scalar multiplication?

The set of all vectors  $(x_1, x_2, x_3, x_4) \in V_4$  such that

- (a)  $x_4 = 0$     (b)  $x_1 = 1$     (c)  $x_2 > 0$   
 (d)  $x_3^2 \geq 0$     (e)  $x_1^2 < 0$     (f)  $2x_1 + 3x_2 = 0$   
 (g)  $x_1 + \frac{2}{3}x_2 - 3x_3 + x_4 = 1$ .

5. In any vector space prove that  $\alpha u = 0$  iff either  $\alpha = 0$  or  $u = 0$ .

6. Which of the following subsets of  $\mathcal{P}$  are vector spaces?

The set of all polynomials  $p$  such that

- (a) degree of  $p \leq n$     (b) degree of  $p = 3$   
 (c) degree of  $p \geq 4$     (d)  $p(1) = 0$   
 (e)  $p(2) = 1$     (f)  $p'(1) = 0$   
 (g)  $p$  has integral coefficients.

7. Which of the following subsets of  $\mathcal{C}[0, 1]$  are vector spaces?

The set of all functions  $f \in \mathcal{C}[0, 1]$  such that

- (a)  $f(1/2) = 0$     (b)  $f(3/4) = 1$   
 (c)  $f'(x) = xf(x)$     (d)  $f(0) = f(1)$   
 (e)  $f(x) = 0$  at a finite number of points in  $[0, 1]$   
 (f)  $f$  has a local maxima at  $x = 1/2$   
 (g)  $f$  has a local extrema at  $x = 1/2$ .

8. Let  $V$  be a real vector space and  $X$  an arbitrary set. Let  $V^X$  be the set of all functions  $f: X \rightarrow V$ . Prove that  $V^X$  is a real vector space for pointwise addition and scalar multiplication, where definitions are analogous to those for real-valued functions.

## 9. True or false?

- (a) In a vector space  $V$ ,  $(-1)(-u) = u$  for all  $u \in V$ .
- (b) In a vector space  $V$ ,  $--u - v = -v - u$  for all  $u, v \in V$ .
- (c) In a vector space  $V$ ,  $--u - v = -(u + v)$  for all  $u, v \in V$ .
- (d) In a vector space  $V$ ,  $-(-u) = u$  for all  $u \in V$ .
- (e)  $R \times R$  is a vector space.
- (f) If the scalars are complex numbers, then  $\mathcal{F}(I)$  is a complex vector space.
- (g) In  $\mathbb{C}$ , the set of complex numbers considered as a complex vector space, if  $\alpha \cdot 1 + \beta i = 0$ , then  $\alpha = 0 = \beta$ .

## 3.2 SUBSPACES

**3.2.1 Definition** Let  $S$  be a nonempty subset of a vector space  $V$ .  $S$  is said to be a *subspace* of  $V$  if  $S$  is a vector space under the same operations of addition and scalar multiplication as in  $V$ .

In order to understand this definition as well as the future concepts in the subject, we shall repeatedly consider the concrete cases of  $V_2$  and  $V_3$ .

$V_2$  is the Euclidean plane. Take any straight line  $S$  through the origin  $O$ . Any point  $P$  on this straight line can be considered as a vector  $\vec{OP}$  of  $V_2$  in  $S$ . The sum of two such vectors  $\vec{OP}$  and  $\vec{OQ}$ , where  $P$  and  $Q$  both lie in  $S$ , is again a vector  $\vec{OR}$ , where  $R$  lies in  $S$ . Similarly, a scalar multiple of any vector in  $S$  is again a vector in  $S$ . All other axioms of a vector space are automatically satisfied in  $S$ . So  $S$  is a vector space under the same operations as in  $V_2$ . Thus,  $S$  is a subspace of  $V_2$ . In other words, every line through the origin is a subspace of  $V_2$ .

In the same manner, in  $V_3$  we find that any plane  $S$  through the origin is a subspace of  $V_3$ . Also, every straight line  $L$  through the origin is a subspace of  $V_3$ . The following examples further illustrate the concept of a subspace.

**Example 3.7** Let  $L$  be the set of all vectors of the form  $(x, 2x, -3x, x)$  in  $V_4$ . Then  $L$  is a subspace of  $V_4$ .

**Proof:** If  $u = (x, 2x, -3x, x)$  and  $v = (y, 2y, -3y, y)$  are two elements of  $L$ , then clearly

$$u + v = (x+y, 2(x+y), -3(x+y), x+y)$$

and

$$au = (ax, 2(ax), -3(ax), ax)$$

also belong to  $L$ . The zero element  $(0, 0, 0, 0)$  is also of this form and hence belongs to  $L$ . The negative of  $u$  is  $-u = (-x, -2x, 3x, -x)$ , which is again of the same form and hence belongs to  $L$ . The other laws of associativity and commutativity for addition, distributive laws, and the final axiom  $1u = u$  are all true in  $L$ , because elements of  $L$  are elements



of  $V_4$ , and in  $V_4$  all these laws are true. Thus,  $L$  is a subspace of  $V_4$ . In fact,  $L$  is the 'line' through the point  $(1, 2, -3, 1)$  and the origin  $(0, 0, 0, 0)$  in  $V_4$ .

**Example 3.8** Generalising Example 3.7, we can say that all scalar multiples of a given element  $u_0$  of a vector space  $V$  form a subspace of  $V$ .

**Proof:** Let the set of all scalar multiples of  $u_0$  be denoted by  $[u_0]$ . If  $u$  and  $v$  are two vectors in  $[u_0]$ , then  $u = \alpha u_0$  for some  $\alpha$  and  $v = \beta u_0$  for some  $\beta$ .

$u + v = \alpha u_0 + \beta u_0 = (\alpha + \beta)u_0$ , i.e.  $u + v$  is a scalar multiple of  $u_0$ . Hence,  $u + v \in [u_0]$ . Again,  $\lambda u = \lambda(\alpha u_0) = (\lambda\alpha)u_0$ , i.e.  $\lambda u$  is also a scalar multiple of  $u_0$ . Hence,  $\lambda u \in [u_0]$  for all scalars  $\lambda$  and  $u \in [u_0]$ .

$0 \in [u_0]$ , because  $0 = 0u_0$  by Theorem 3.1.7. If  $u \in [u_0]$ , then  $-u = (-1)u$  by Theorem 3.1.7, and so  $-u \in [u_0]$ . The other axioms, which are only interrelations, are true for all elements of  $V$  and so are true for all elements of  $[u_0]$ . Hence,  $[u_0]$  is a subspace of  $V$ .

Example 3.7 is a special case of Example 3.8. The subspace of  $V_4$  considered in Example 3.7 is just  $\{(1, 2, -3, 1)\}$ .

In these examples we note that to prove that  $S$  is a subspace of  $V$  we explicitly checked only the following :

(i) The sum of any two vectors in  $S$  is again in  $S$ , i.e. addition is closed in  $S$ .

(ii) The scalar multiple of any vector in  $S$  is again in  $S$ , i.e. scalar multiplication is closed in  $S$ .

(iii) The existence of  $0$  in  $S$  and the existence of a negative for each element in  $S$ .

The other axioms were not explicitly checked, because this, as the following theorem shows, was not necessary. In fact, the theorem says that even (iii) need not have been checked.

**3.2.2 Theorem** A nonempty subset  $S$  of a vector space  $V$  is a subspace of  $V$  iff the following conditions are satisfied :

(a) If  $u, v \in S$ , then  $u + v \in S$ .

(b) If  $u \in S$  and  $\alpha$  a scalar, then  $\alpha u \in S$ .

In other words, a subset  $S$  of a vector space  $V$  is a subspace of  $V$  iff it is closed under addition and scalar multiplication defined in  $V$ .

**Proof:** Let  $S$  be a subspace of  $V$ . Then  $S$  is a vector space under the same operations as those of  $V$ . Hence,  $S$  satisfies (a) and (b). Conversely, if (a) and (b) are satisfied, then we have to prove that  $S$  satisfies all the axioms of a vector space. VS1 and VS2 are satisfied for  $S$ , because this is exactly what (a) and (b) say. We shall now show that VS3 for  $S$  follows from VS1 and VS2 for  $S$ .

$0u = 0$  for any  $u \in V$  and therefore for any  $u_0 \in S$ . Taking any  $u_0 \in S$ , we see that, from (b),  $0 = 0u_0 \in S$ . Similarly,  $(-1)u = -u$  for any  $u \in V$  and therefore also for a given  $u_0 \in S$ . Hence,  $-u_0 \in S$  for every  $u_0 \in S$ . Thus, G2 and G3 hold in  $S$ .

Axioms G1, G4, and (b), (c), and (d) of VS3 are automatically satisfied for elements of  $S$ , since these laws already hold for elements of  $V$ . Note that this needs no checking, because these axioms are interrelations of the elements. In the cases of G2 and G3 the checking was necessary, because we had to show the existence of certain elements in  $S$ . ■

**3.2.3 Remark** In view of Theorem 3.2.2, whenever we want to prove that a set  $S$  for certain operations is a vector space, we try to look at  $S$  as a subset of  $V$ , where  $V$  is a vector space and the operations of addition and scalar multiplication in  $S$  are the same as the operations in  $V$ . Once we have done this, it is enough to prove that the operations are closed in  $S$ .

*Example 3.9* Take  $\mathcal{C}[a, b]$  of Example 3.5.  $\mathcal{C}[a, b]$  is a subset of  $\mathcal{F}[a, b]$ , which is a vector space by Example 3.2. So to prove that  $\mathcal{C}[a, b]$  is a vector space (which we have done in Example 3.5), it is enough to prove that  $\mathcal{C}[a, b]$  is a subspace of  $\mathcal{F}[a, b]$ . Therefore, by Theorem 3.2.2, we only need to prove that  $\mathcal{C}[a, b]$  is closed for addition and scalar multiplication.

Since the sum of two continuous functions is continuous and any scalar multiple of a continuous function is again continuous, we find that addition and scalar multiplication are closed in  $\mathcal{C}[a, b]$ . This observation not only proves that  $\mathcal{C}[a, b]$  is a vector space, but also that it is a subspace of  $\mathcal{F}[a, b]$ .

**3.2.4 Remark** The spaces  $\mathcal{H}[a, b]$ ,  $\mathcal{C}^{(1)}[a, b]$ ,  $\mathcal{C}^{(n)}[a, b]$ , and  $\mathcal{P}[a, b]$  are subspaces of  $\mathcal{F}[a, b]$ .

Further, note that

- (a)  $\mathcal{P}[a, b]$  is a subspace of  $\mathcal{C}[a, b]$ .
- (b)  $\mathcal{C}^{(1)}[a, b]$  is a subspace of  $\mathcal{C}[a, b]$ .
- (c)  $\mathcal{C}^{(n)}[a, b]$  is a subspace of  $\mathcal{C}[a, b]$  for every positive integer  $n$ .
- (d)  $\mathcal{C}^{(n)}[a, b]$  is a subspace of  $\mathcal{C}^{(m)}[a, b]$  for every  $m < n$ .
- (e)  $\mathcal{I}[a, b]$  is a subspace of  $\mathcal{C}^{(n)}[a, b]$  for every positive integer  $n$ .
- (f) Similar results are true for functions defined on  $(a, b)$ .

*Example 3.10* The set  $S$  of all polynomials  $p \in \mathcal{P}$ , which vanishes at a fixed point  $x_0$ , is a subspace of  $\mathcal{P}$ .

We have  $S = \{p \in \mathcal{P} \mid p(x_0) = 0\}$ .

If  $p$  and  $q$  are two members of  $S$ , then  $p(x_0) = 0$  and  $q(x_0) = 0$ . So  $(p + q)(x_0) = p(x_0) + q(x_0) = 0$ , which means that the polynomial  $p + q$  also vanishes at  $x_0$  and so  $p + q \in S$ . So addition is closed in  $S$ .

Similarly, if  $\alpha$  is a scalar and  $p \in S$ , then  $(\alpha p)(x_0) = \alpha(p(x_0)) = \alpha \cdot 0 = 0$ . So  $\alpha p \in S$ . This completes the proof that  $S$  is a subspace of  $\mathcal{P}$ .

Before concluding this article, we shall identify what are called trivial subspaces of a vector space  $V$ . The set containing just the zero element of  $V$  and nothing else is clearly a subspace, since it satisfies conditions (a) and (b) of Theorem 3.2.2. Similarly,  $V$ , itself considered as a subset of  $V$ , satisfies the conditions of Theorem 3.2.2 and so is a subspace of itself. These two subspaces  $\{0\}$  and  $V$  of  $V$  are called *trivial subspaces* of  $V$ , since they trivially satisfy conditions (a) and (b). All other subspaces of  $V$  are called *nontrivial subspaces* of  $V$ . The trivial subspace  $\{0\}$  of  $V$  is denoted by  $V_0$  and is also called the zero subspace of  $V$ .

**Example 3.11** Consider the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0. \quad (1)$$

where  $\alpha_i$ 's are real constants and  $x_i$ 's are real unknowns. A solution of this equation can be represented as an  $n$ -tuple  $(x_1, x_2, \dots, x_n)$ , which is a vector of  $V_n$ . Now let  $S$  be the set of all vectors  $(x_1, x_2, \dots, x_n) \in V_n$ , which satisfy Equation (1). We can prove that  $S$  is a subspace of  $V_n$ . The proof is left as an exercise for the reader.

The special cases of Example 3.1, when  $n = 2$  or  $n = 3$ , are interesting. Take  $V_2$ . The set  $S$  of vectors  $(x, y) \in V_2$ , which satisfy the equation  $\alpha x + \beta y = 0$ , is clearly a straight line through the origin in  $V_2$ . It is therefore a subspace of  $V_2$ , as pointed out in the discussion following Definition 3.2.1. This is now corroborated by the general case considered in Example 3.11.

Again, in  $V_3$  the set  $S$  of all vectors  $(x, y, z) \in V_3$ , which satisfy the equation  $\alpha x + \beta y + \gamma z = 0$ , is a plane through the origin and hence a subspace of  $V_3$  (also explained in the discussion following Definition 3.2.1). This fact is now corroborated by the result in the general case.

Finally, suppose in  $V_3$  we consider the set  $S$  of all vectors  $(x, y, z)$ , which satisfy the equation  $\alpha x + \beta y + \gamma z = 1$ . This, of course, is a plane, but since it does not contain the vector  $(0, 0, 0)$ , the zero of the space, it is not a subspace.

From the foregoing argument we can see that to prove that a given set is not a subspace (when it is not!) is really easy. For, from among the several axioms that it has to satisfy we choose one that is not satisfied by it and we are done. For example, in the previous paragraph, it is obvious that  $(0, 0, 0)$  does not belong to the set and this observation alone is enough for us to conclude that the set is not a subspace.

### Problem Set 3.2

1. Prove that conditions (a) and (b) of Theorem 3.2.2 can be replaced by the single condition

$\alpha u + \beta v \in S$  for all  $u, v \in S$  and all scalars  $\alpha, \beta$ .

2. Let  $W = \{(x_1, x_2, \dots, x_n) \in V_n \mid x_1 = 0\}$ . Prove that  $W$  is a subspace of  $V_n$ .

3. Prove that

$$W = \{(x_1, x_2, \dots, x_n) \in V_n^C \mid \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0, \\ \alpha_i \text{'s are given constants}\}$$

is a subspace of  $V_n^C$ .

4. Which of the following sets are subspaces of  $V_3$ ?

- (a)  $\{(x_1, x_2, x_3) \mid x_1 x_2 = 0\}$
- (b)  $\{(x_1, x_2, x_3) \mid \frac{x_2}{x_1} = \sqrt{2}\}$
- (c)  $\{(x_1, x_2, x_3) \mid \sqrt{2}x_1 = \sqrt{3}x_2\}$
- (d)  $\{(x_1, x_2, x_3) \mid x_3 \text{ is an integer}\}$
- (e)  $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 < 1\}$
- (f)  $\{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 \geq 0\}$
- (g)  $\{(x_1, x_2, x_3) \mid x_1 = \sqrt{2}x_2 \text{ and } x_1 = 3x_3\}$
- (h)  $\{(x_1, x_2, x_3) \mid x_1 = 2x_2 \text{ or } x_3 = 3x_2\}$
- (i)  $\{(x_1, x_2, x_3) \mid x_1 - 2x_2 = x_3 - \frac{3x_2}{2}\}$ .

5. Which of the following sets are subspaces of  $\mathcal{P}$ ?

- (a)  $\{p \in \mathcal{P} \mid \text{degree of } p = 4\}$
- (b)  $\{p \in \mathcal{P} \mid \text{degree of } p \leq 3\}$
- (c)  $\{p \in \mathcal{P} \mid \text{degree of } p \geq 5\}$
- (d)  $\{p \in \mathcal{P} \mid \text{degree of } p \leq 4 \text{ and } p'(0) = 0\}$
- (e)  $\{p \in \mathcal{P} \mid p(1) = 0\}$ .

6. Which of the following sets are subspaces of  $\mathcal{C}^2(a, b)$ ?

- (a)  $\{f \in \mathcal{C}^2(a, b) \mid f(x_0) = 0, x_0 \in (a, b)\}$
- (b)  $\{f \in \mathcal{C}^2(a, b) \mid f'(x) = 0 \text{ for all } x \in (a, b)\}$
- (c)  $\{f \in \mathcal{C}^2(a, b) \mid f(\frac{a+b}{2}) = 1\}$
- (d)  $\{f \in \mathcal{C}^2(a, b) \mid f'(x) = x^2 f(x)\}$
- (e)  $\{f \in \mathcal{C}^2(a, b) \mid 2f'''(x) + 3xf''(x) - f'(x) + x^2 f(x) = 0\}$
- (f)  $\{f \in \mathcal{C}^2(a, b) \mid \int_a^b f(x) dx = 0\}$ .

7. Prove the statements in Remark 3.2.4.

8. (a) If  $U$  and  $W$  are subspaces of a vector space  $V$  such that  $U \subset W$ , then prove that  $U$  is a subspace of  $W$ .

- (b) If  $W$  is a subspace of a vector space  $V$  and  $U$  is a subspace of  $W$ , then prove that  $U$  is a subspace of  $V$ .

9. True or false?

- (a) In  $V_1$  every infinite subset is a subspace.

- (b)  $V_2$  is a subspace of  $V_3$ .
- (c)  $V_1 \times V_2 = V_3$ .
- (d) Every subspace of a vector space is a vector space.
- (e)  $V_0$  is the only real vector space containing a finite number of elements.
- (f) The derivatives of  $e^x$  form a vector space.
- (g) The set  $\{\alpha \sin x + \beta \cos x \mid \alpha, \beta \text{ are real numbers}\}$  is a vector space.

### 3.3 SPAN OF A SET

In Example 3.8 we saw that an element  $u_0$  of a vector space  $V$  gave rise to a subspace  $[u_0]$  of  $V$ . Looking at it another way, we can say that here is a subspace of  $V$ , namely,  $[u_0]$ , that is fully identified once we know the single element  $u_0$ . In this article we shall see subspaces that are fully identified by a subset of  $V$ , which is smaller than the subspace. To be more precise, we need the following definitions.

**3.3.1 Definition** Let  $u_1, u_2, \dots, u_n$  be  $n$  vectors of a vector space  $V$  and let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be  $n$  scalars. Then

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$$

is called a *linear combination* of  $u_1, u_2, \dots, u_n$ . It is also called a linear combination of the set  $S = \{u_1, u_2, \dots, u_n\}$ . This being a linear combination of a finite set is also called a *finite linear combination*.

In the case of an infinite set  $S$  a linear combination of a finite subset  $A$  of  $S$  is referred to as a *finite linear combination* of  $S$ .

**3.3.2 Definition** The *span* of a subset  $S$  of a vector space  $V$  is the set of all finite linear combinations of  $S$ .

In other words, if  $S$  is a subset of  $V$ , the span of  $S$  is the set

$$\left\{ \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \mid \begin{array}{l} \alpha_1, \alpha_2, \dots, \alpha_n \text{ any scalars, } n \text{ any posi-} \\ \text{tive integer, and } u_1, u_2, \dots, u_n \in S \end{array} \right\}.$$

The span of  $S$  is denoted by  $[S]$ . If  $S$  contains only a finite number of elements, say  $u_1, u_2, \dots, u_n$ , then  $[S]$  is also written as  $[u_1, u_2, \dots, u_n]$ .

As illustration, take  $V = V_3$  and  $S = \{(1, 0, 0), (0, 1, 0)\}$ . Any linear combination of a finite number of elements of  $S$  is of the form  $\alpha(1, 0, 0) + \beta(0, 1, 0) = (\alpha, \beta, 0)$ . The set of all such linear combinations is  $[S]$ . Actually,  $[S] = \{(\alpha, \beta, 0) \mid \alpha, \beta \text{ any scalars}\}$ . In this case  $[S]$  can be written also as  $[(1, 0, 0), (0, 1, 0)]$ . We can see that it is a subspace of  $V_3$ . In fact, this is true in all cases. We shall state and prove this assertion in the form of a theorem.

**3.3.3 Theorem** Let  $S$  be a nonempty subset of a vector space  $V$ . Then  $[S]$ , the span of  $S$ , is a subspace of  $V$ .

**Proof:** By Theorem 3.2.2, we have only to prove that  $[S]$  is closed for addition and scalar multiplication. Let  $u$  and  $v$  be any two vectors in  $[S]$ . Then

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \text{ for some scalars } \alpha_i, \text{ some } u_i\text{'s} \in S, \text{ and a positive integer } n,$$

and  $v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m$  for some scalars  $\beta_i$ , some vectors  $v_i$ 's  $\in S$ , and a positive integer  $m$ .

Hence,  $u + v = \alpha_1 u_1 + \dots + \alpha_n u_n + \beta_1 v_1 + \dots + \beta_m v_m$ . This is again a finite linear combination of  $S$  and so  $u + v \in [S]$ . Similarly,  $\alpha u = (\alpha \alpha_1) u_1 + \dots + (\alpha \alpha_n) u_n$  is again a finite linear combination of  $S$  and so it is in  $[S]$ . Hence,  $[S]$  is a subspace of  $V$ . ■

**3.3.4 Remark** A nontrivial subspace always contains an infinite number of elements. So  $[S]$  ( $\neq V_0$ ) always contains an infinite number of elements. But  $S$  itself may be a smaller set, even a finite set. By convention, we take  $[\phi] = V_0$ .

**3.3.5 Theorem** If  $S$  is a nonempty subset of a vector space  $V$ , then  $[S]$  is the smallest subspace of  $V$  containing  $S$ .

**Proof:** Clearly,  $[S]$  is a subspace by Theorem 3.3.3. It contains  $S$ , because each element  $u_0$  of  $S$  can be written as  $1u_0$ , i.e. a finite linear combination of  $S$ . To prove that  $[S]$  is the smallest subspace containing  $S$ , we shall show that if there exists another subspace  $T$  containing  $S$ , then  $T$  contains  $[S]$  also.

So let a subspace  $T$  contain  $S$ . We have to prove that  $T$  contains  $[S]$ . Take any element of  $[S]$ . It is of the form  $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$ , where  $\alpha_i$ 's are scalars,  $u_1, u_2, \dots, u_n$  are in  $S$ , and  $n$  is a positive integer. Since  $S \subset T$ , each  $u_i$  also belongs to  $T$ . Since  $T$  is a subspace,  $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n$  should also belong to  $T$ . This means that each element of  $[S]$  is in  $T$ . ■

**Example 3.12** In  $V_2$  show that  $(3, 7)$  belongs to  $[(1, 2), (0, 1)]$ , but does not belong to  $[(1, 2), (2, 4)]$ .

Clearly,  $(3, 7)$  belongs to  $[(1, 2), (0, 1)]$ , if it is a linear combination of  $(1, 2)$  and  $(0, 1)$ , i.e. if

$$(3, 7) = \alpha(1, 2) + \beta(0, 1) = (\alpha, 2\alpha + \beta)$$

for some suitable  $\alpha$  and  $\beta$ . This is possible only if

$$\alpha = 3, \quad 2\alpha + \beta = 7. \quad (1)$$

Solving Equation (1), we get  $\alpha = 3$ ,  $\beta = 1$ . Thus,  $(3, 7) = 3(1, 2) + 1(0, 1)$ . Hence,  $(3, 7) \in [(1, 2), (0, 1)]$ .

Further, if  $(3, 7) \in [(1, 2), (2, 4)]$ , then

$$(3, 7) = \alpha(1, 2) + \beta(2, 4) = (\alpha + 2\beta, 2\alpha + 4\beta).$$

for some suitable  $\alpha$  and  $\beta$ . This gives

$$\alpha + 2\beta = 3, \quad 2\alpha + 4\beta = 7. \quad (2)$$

But these equations cannot hold at the same time. So  $(3, 7) \notin [(1, 2), (2, 4)]$ .

**Example 3.13** In the complex vector space  $V_{\mathbb{C}}^2$  show that  $(1 + i, 1 - i)$  belongs to  $[(1 + i, 1), (1, 1 - i)]$ .

$$\begin{aligned} [(1 + i, 1), (1, 1 - i)] &= \{\alpha(1 + i, 1) + \beta(1, 1 - i) \mid \alpha, \beta \text{ complex scalars}\} \\ &= \{(\alpha + \beta + \alpha i, \alpha + \beta - \beta i) \mid \alpha, \beta \text{ complex scalars}\}. \end{aligned}$$

$(1 + i, 1 - i)$  belongs to  $[(1 + i, 1), (1, 1 - i)]$  if

$$(1 + i, 1 - i) = (\alpha + \beta + \alpha i, \alpha + \beta - \beta i)$$

for some scalars  $\alpha, \beta$ . Thus,

$$\begin{aligned} 1 + i &= \alpha + \beta + \alpha i = \alpha(1 + i) + \beta \\ 1 - i &= \alpha + \beta - \beta i = \alpha + \beta(1 - i). \end{aligned}$$

Solving for  $\alpha, \beta$ , we get  $\alpha = 1 + i, \beta = 1 - i$ , showing that  $(1 + i, 1 - i)$  belongs to  $[(1 + i, 1), (1, 1 - i)]$ .

### Problem Set 3.3

- Let  $S = \{(1, 2, 1), (1, 1, -1), (4, 5, -2)\}$ . Determine which of the following vectors are in  $[S]$ :
  - $(0, 0, 0)$
  - $(1, 1, 0)$
  - $(2, -1, -8)$
  - $(-\frac{1}{3}, -\frac{1}{3}, \frac{1}{3})$
  - $(1, 0, 1)$
  - $(1, -3, 5)$
- Let  $S = \{x^3, x^2 + 2x, x^2 + 2, 1 - x\}$ . Determine which of the following polynomials are in  $[S]$ :
  - $2x^3 + 3x^2 + 3x + 7$
  - $x^4 + 7x + 2$
  - $3x^2 + x + 5$
  - $x^3 - \frac{3}{2}x^2 + \frac{x}{2}$
  - $3x + 2$
  - $x^3 + x^2 + 2x + 3$
- If  $S$  is a nonempty subset of a vector space  $V$ , prove that
  - $[S] = S$  iff  $S$  is a subspace of  $V$
  - $[[S]] = [S]$ .
- Let  $v_1, v_2, \dots, v_n$  be  $n$  elements of a vector space  $V$ . Then prove that
  - $[v_1, v_2, \dots, v_n] = [\alpha_1 v_1, \alpha_2 v_2, \dots, \alpha_n v_n], \alpha_i \neq 0$
  - $[v_1, v_2] = [v_1 - v_2, v_1 + v_2]$
  - If  $v_k \in [v_1, v_2, \dots, v_{k-1}]$ , then  $[v_1, v_2, \dots, v_{k-1}, v_k, v_{k+1}, \dots, v_n] = [v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_n]$ .
- Let  $S$  be a nonempty subset of a vector space  $V$  and  $u, v \in V$ . If  $u \in [S \cup \{v\}]$  but  $u \notin [S]$ , then prove that  $v \in [S \cup \{u\}]$ .
- What is the span of
  - $x$ -axis and  $y$ -axis in  $V_3$ ?
  - $x$ -axis and  $xy$ -plane in  $V_3$ ?

- (c)  $xy$ -plane and  $yz$ -plane in  $V_3$  ?  
 (d)  $x$ -axis and the plane  $x + y = 0$  in  $V_3$  ?
7. True or false ?
- (a) Span of  $x + y = 0$  and  $x - y = 0$  in  $V_3$  is  $V_3$ .  
 (b)  $V_7$  has a subspace consisting of 7 elements.  
 (c) The span of the set  $\{x, x^2, x^3, x^4, \dots\}$  is  $\mathcal{P}$ .  
 (d) In  $V_3$ ,  $[y\text{-axis} \cup z\text{-axis}] = V_3$ .  
 (e) In  $V_3$ ,  $[k] = [i, k, i + k] \cap [j, k, j + k]$ .  
 (f) In  $V_3^C$ ,  $(-1, 2, 0) \in [(i, -i, 1), (0, -i, -1)]$ .

### 3.4 MORE ABOUT SUBSPACES

Let  $U$  and  $W$  be two subspaces of a vector space  $V$ . Their intersection  $U \cap W$  cannot be empty, because each contains the zero vector of  $V$ .

Now, if  $u$  and  $v$  are two vectors of  $U \cap W$ , then  $u + v \in U$  and  $u + v \in W$ , because  $U$  and  $W$  are subspaces of  $V$  and  $u, v$  both belong to  $U$  as well as  $W$ . Hence,  $u + v \in U \cap W$ . Similarly, if  $\alpha$  is any scalar, then  $\alpha u$  is in both  $U$  and  $W$ . Hence,  $\alpha u \in U \cap W$ . This shows that if  $U$  and  $W$  are subspaces of  $V$ , then  $U \cap W$  is also a subspace of  $V$ .

This result can be generalised to any number of subspaces. More precisely, if  $U_1, U_2, \dots, U_n$  are  $n$  subspaces of  $V$ , then their intersection  $U_1 \cap U_2 \cap \dots \cap U_n$  is also a subspace of  $V$  (see Problem 1). We shall use this fact in an interesting way in the following example.

**Example 3.14** Let  $W$  be the set of all vectors  $(x_1, x_2, \dots, x_n)$  of  $V_n$  satisfying the three equations

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0 \quad (1)$$

$$\beta_1 x_1 + \beta_2 x_2 + \dots + \beta_n x_n = 0 \quad (2)$$

$$\gamma_1 x_1 + \gamma_2 x_2 + \dots + \gamma_n x_n = 0. \quad (3)$$

Then  $W = W_1 \cap W_2 \cap W_3$ , where  $W_1$  is the solution set of Equation (1),  $W_2$  is the solution set of Equation (2), and  $W_3$  is the solution set of Equation (3). Each  $W_i$  is a subspace by Example 3.11. So, by the foregoing argument,  $W$  is a subspace of  $V_n$ .

Thus, we have proved that the intersection of two subspaces is always a subspace. On the other hand, the union of two subspaces need not be a subspace. For example, take  $U = x$ -axis and  $W = y$ -axis in  $V_2$ .  $U$  and  $W$  are subspaces of  $V_2$ . Here  $(1, 0) \in U$  and  $(0, 1) \in W$ . So both  $(1, 0)$  and  $(0, 1)$  belong to  $U \cup W$ . But  $(1, 0) + (0, 1) = (1, 1) \notin U \cup W$ . This shows that  $U \cup W$  is not a subspace of  $V_2$ .

Thus,  $U \cup W$  is not in general a subspace. However, we know, by Theorem 3.3.5, that  $[U \cup W]$  is the smallest subspace of  $V$  containing



$U \cup W$ . Let us now analyse the subspace  $[U \cup W]$ . Any element of this subspace is a linear combination of a finite subset of  $U \cup W$ . In other words, a vector  $v$  of  $[U \cup W]$  is of the form

$$v = \alpha_1 u_1 + \dots + \alpha_n u_n + \beta_1 w_1 + \dots + \beta_m w_m, \quad (4)$$

where  $u_i$ 's  $\in U$ ,  $w_i$ 's  $\in W$ ,  $n$  and  $m$  are nonnegative integers, and  $\alpha$ 's,  $\beta$ 's are scalars. But  $\alpha_1 u_1 + \dots + \alpha_n u_n$  is a vector in  $U$  and  $\beta_1 w_1 + \dots + \beta_m w_m$  is a vector in  $W$ . Therefore,  $v$  can be expressed as  $u + w$ , where  $u \in U$  and  $w \in W$ . So we can say that  $[U \cup W]$  consists of elements of the form  $u + w$ ,  $u \in U$ , and  $w \in W$ .

**3.4.1 Definition (Addition of sets)** Let  $A$  and  $B$  be two subsets of a vector space  $V$ . Then the sum of  $A$  and  $B$ , written as  $A + B$ , is the set of all vectors of the form  $u + v$ ,  $u \in A$ , and  $v \in B$ , that is,

$$A + B = \{u + v \mid u \in A, v \in B\}.$$

**Example 3.15** In  $V_2$  let  $A = \{(1, 2), (0, 1)\}$  and  $B = \{(1, 1), (-1, 2), (2, 5)\}$ . Then

$$\begin{aligned} A + B &= \{(1, 2) + (1, 1), (1, 2) + (-1, 2), (1, 2) + (2, 5), (0, 1) \\ &\quad + (1, 1), (0, 1) + (-1, 2), (0, 1) + (2, 5)\} \\ &= \{(2, 3), (0, 4), (3, 7), (1, 2), (-1, 3), (2, 6)\}. \end{aligned}$$

**Example 3.16** In  $V_2$  let  $A = \{(2, 3)\}$ ,  $B = \{t(3, 1) \mid t \text{ a scalar}\}$ . Then

$$\begin{aligned} A + B &= \{(2, 3) + t(3, 1) \mid t \text{ a scalar}\} \\ &= \{(2 + 3t, 3 + t) \mid t \text{ a scalar}\}. \end{aligned}$$

Geometrically,  $B$  is a line through the origin and  $A$  is a set containing one vector (Figure 3.1).  $A + B$  is a line parallel to  $B$  and passing through the point  $(2, 3)$ .

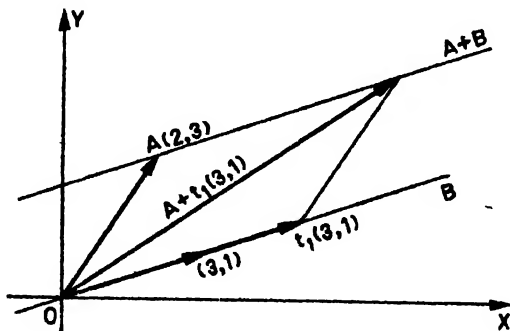


FIGURE 3.1

**Example 3.17** In  $V_3$  let  $A = \{\alpha(1, 2, 0) \mid \alpha \text{ a scalar}\}$ ,  $B = \{\beta(0, 1, 2) \mid \beta \text{ a scalar}\}$ . Then

$$\begin{aligned} A + B &= \{\alpha(1, 2, 0) + \beta(0, 1, 2) \mid \alpha, \beta \text{ scalars}\} \\ &= \{(\alpha, 2\alpha, +\beta, 2\beta) \mid \alpha, \beta \text{ scalars}\}. \end{aligned}$$

Geometrically,  $A$  and  $B$  are lines through the origin in  $V_3$  and  $A + B$  is a plane containing these lines and passing through the origin (Figure 3.2).

In this case  $A + B = [(1, 2, 0), (0, 1, 2)]$ .

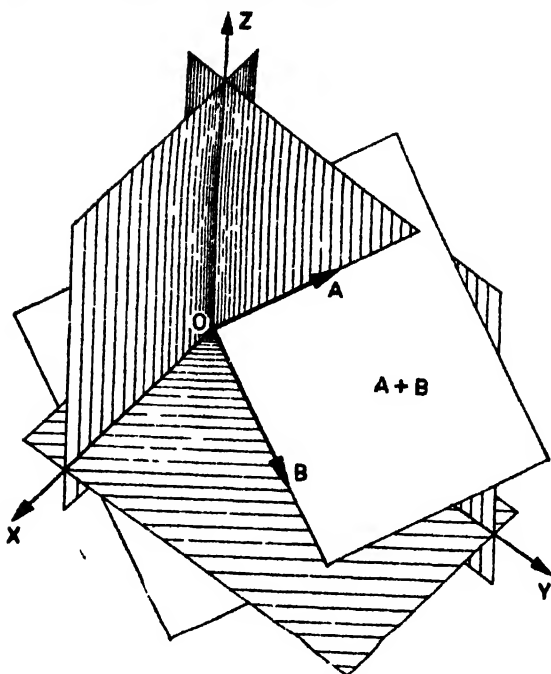


FIGURE 3.2

**3.4.2 Theorem** If  $U$  and  $W$  are two subspaces of a vector space  $V$ , then  $U + W$  is a subspace of  $V$  and  $U + W = [U \cup W]$ .

*Proof:* Obviously,  $U + W \subset [U \cup W]$ , because each vector of  $U + W$  is a finite linear combination of  $U \cup W$ . We shall now prove that  $[U \cup W] \subset U + W$ . Let  $v \in [U \cup W]$ . Then the argument preceding Definition 3.4.1 shows that  $v$  is of the form  $u + w$ , for some  $u \in U$  and  $w \in W$ . Hence,  $v \in U + W$ . Thus,  $U + W = [U \cup W]$ . This automatically makes  $U + W$  a subspace of  $V$ . ■

**3.4.3 Remark** From Theorem 3.4.2 it follows that  $U + W$  is the smallest subspace of  $V$  containing  $U \cup W$ , i.e. both  $U$  and  $W$ .

**Example 3.18** If  $V = V_3$ ,  $U = x$ -axis, and  $W = y$ -axis, then  $U + W$  is the set of all those vectors of  $V_3$  that are of the form  $\alpha(1, 0, 0) + \beta(0, 1, 0)$ . Therefore,  $U + W = \{(\alpha, \beta, 0) \mid \alpha, \beta \text{ are scalars}\}$ . On the other hand,  $[U \cup W]$  consists of vectors of the form  $\alpha u + \beta w$ , where  $\alpha, \beta$  are scalars,  $u \in U, w \in W$ . This gives

$$[U \cup W] = \{\alpha(1, 0, 0) + \beta(0, 1, 0) \mid \alpha, \beta \text{ are scalars}\},$$

which is the same as  $U + W$ . This illustrates Theorem 3.4.2. Note the interesting relation arising from this example :

$$[x\text{-axis} \cup y\text{-axis}] = x\text{-axis} + y\text{-axis} = xy\text{-plane.} \quad (5)$$

## DIRECT SUM

We have just seen that if  $U$  and  $W$  are subspaces of a vector space  $V$ , then the sum  $U + W$  is also a subspace of  $V$ . If, in addition,  $U \cap W = V_0 = \{0\}$ , the sum  $U + W$  is called a *direct sum*. The direct sum of  $U$  and  $W$  is written as  $U \oplus W$ .

**Example 3.19** We can check the following additions in  $V_3$  :

$$xy\text{-plane} + yz\text{-plane} = V_3 \quad (6)$$

$$xy\text{-plane} + z\text{-axis} = V_3. \quad (7)$$

The sum in Equation (7) is a direct sum, because  $(xy\text{-plane}) \cap z\text{-axis} = \{0\}$ . So Equation (7) can be rewritten as

$$(xy\text{-plane}) \oplus (z\text{-axis}) = V_3. \quad (8)$$

The sum in Equation (6) is not a direct sum, because

$$(xy\text{-plane}) \cap (yz\text{-plane}) = (y\text{-axis}) \neq \{0\}.$$

Any vector  $(a, b, c) \in V_3$  can be written as

$$(a, b, c) = (a, b, 0) + (0, 0, c), \quad (9)$$

where  $(a, b, c) \in xy\text{-plane}$ ,  $(0, 0, c) \in z\text{-axis}$ . Thus,  $(a, b, c)$  is the sum of two vectors, one in the  $xy\text{-plane}$  and the other in the  $z\text{-axis}$ . The advantage of the direct sum lies in the fact that the representation (9) is unique. That is, we cannot find two other vectors, one in the  $xy\text{-plane}$  and the other in  $z\text{-axis}$ , whose sum is  $(a, b, c)$ . The reader can check this for himself.

On the other hand, in Equation (6) any vector  $(a, b, c)$  can be written as the sum of two vectors, one in the  $xy\text{-plane}$  and the other in the  $yz\text{-plane}$ , in more than one way, for example,

$$(a, b, c) = (a, b, 0) + (0, 0, c) \quad (10)$$

and

$$(a, b, c) = (a, 0, 0) + (0, b, c). \quad (11)$$

The fact asserted in this example is generalised in the following theorem.

**3.4.4 Theorem** Let  $U$  and  $W$  be two subspaces of a vector space  $V$  and  $Z = U + W$ . Then  $Z = U \oplus W$  iff the following condition is satisfied :

Any vector  $z \in Z$  can be expressed uniquely as the sum

$$z = u + w, \quad u \in U, w \in W. \quad (12)$$

**Proof:** Let  $Z = U \oplus W$ . Since  $Z = U + W$ , any vector  $z \in Z$  can be written as

$$z = u + w, \quad (13)$$

for some  $u \in U$  and  $w \in W$ .

Suppose it is possible to have another representation

$$z = u' + w', \quad (14)$$

for some  $u' \in U$  and  $w' \in W$ . Then

$$u + w = u' + w' \quad (14)$$

or

$$u - u' = w' - w. \quad (15)$$

But  $u - u' \in U$  and  $w' - w \in W$ . From Equation (15), we find that  $u - u' = w' - w \in U \cap W = \{0\}$ , since  $Z$  is a direct sum of  $U$  and  $W$ . Hence,  $u - u' = 0 = w' - w$ , which implies that  $u = u'$  and  $w = w'$ . So no such second representation for Equation (13) is possible. In other words, condition (12) is satisfied.

Conversely, let condition (12) hold. We shall now prove that  $Z$  is a direct sum of  $U$  and  $W$ . Since  $Z = U + W$ , we have only to prove that  $U \cap W = \{0\}$ .

Let, if possible,  $U \cap W$  contain a nonzero vector  $v$ . Then  $v \in U$ ,  $v \in W$ , and  $v = v + 0 \in U + W$  with  $v \in U$  and  $0 \in W$ . Also  $v = 0 + v \in U + W$  with  $0 \in U$  and  $v \in W$ . These two ways of expressing a vector  $v$  in  $U + W$  contradict the hypothesis (12). Hence,  $U \cap W = \{0\}$  and  $Z = U \oplus W$ . ■

**3.4.5 Definition** If  $U$  is a subspace of a vector space  $V$  and  $v$  a vector of  $V$ , then  $\{v\} + U$ , also written as  $v + U$ , is called a *translate of  $U$*  (by  $v$ ) or a *parallel of  $U$*  (through  $v$ ) or a *linear variety*.

$U$  is called the *base space* of the linear variety and  $v$  a *leader*. Note that  $\{v\} + U$  is not a subspace unless  $v \in U$ . (Why?)

As illustration, take the line  $y = x$  through the origin in  $V_2$ . Call it  $U$ . Consider the point  $v = (1, 0)$ . The translate  $v + U$  of  $U$  by  $v$  is the line  $y = x - 1$  through the point  $(1, 0)$  (Figure 3.3). It can also be obtained by adding  $(1, 0)$  to the vectors in  $y = x$ .

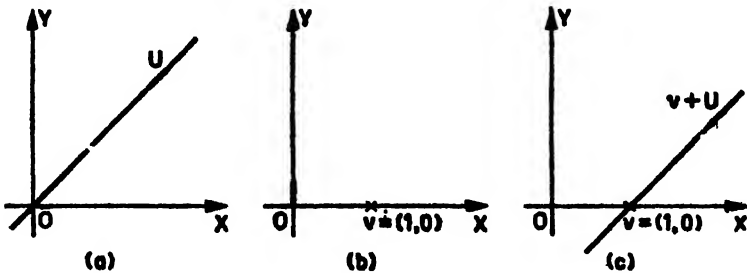


FIGURE 3.3

As a second illustration, take the plane  $y = 0$  in  $V_3$  and call it  $U$ . Consider the point  $v = (1, 1, 1) \in V_3$ .  $(1, 1, 1) + U$  is the set of all points of

$V_3$  given by  $(1, 1, 1) + u$ , where  $u \in U$ . Geometrically, it is the plane parallel to  $y = 0$  through the point  $(1, 1, 1)$  (Figure 3.4).

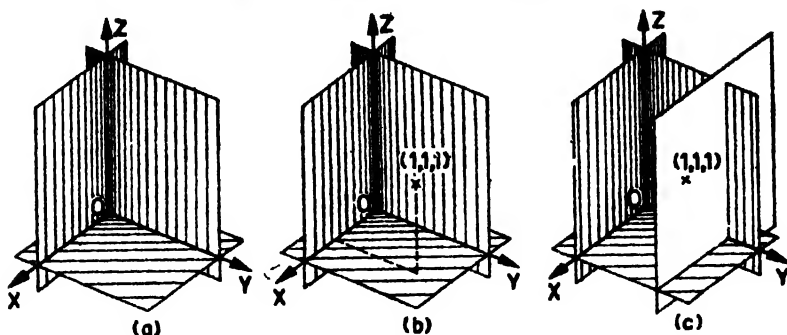


FIGURE 3.4

The following theorem gives the fundamental properties of a linear variety.

**3.4.6 Theorem** Let  $U$  be a subspace of a vector space  $V$  and  $P = v + U$  be the parallel of  $U$  through  $v$ . Then

(a) for any  $w$  in  $P$ ,  $w + U = P$ . In other words, any vector of  $P$  can be taken as a leader of  $P$ ;

(b) two vectors  $v_1, v_2 \in V$  are in the same parallel of  $U$  iff  $v_1 - v_2 \in U$ .

*Proof:* (a) Let  $w$  be a vector of  $P$ . Since  $P = v + U$ ,  $w$  can be expressed as  $v + u_1$  for some  $u_1 \in U$ . So  $w = v + u_1$  or, equivalently,  $v = w - u_1$ .

Let  $z \in P$ . Then  $z$  has the expression  $z = v + u_2$  for a suitable  $u_2 \in U$ . Since  $v = w - u_1$ , we have  $z = (w - u_1) + u_2 = w + (u_2 - u_1)$ . Here  $u_2 - u_1 \in U$ . Thus, every vector  $z$  of  $P$  has the form  $w + (\text{some vector in } U)$ . So  $P \subset w + U$ . Conversely, let  $y \in w + U$ . Then  $y$  is of the form  $y = w + u$  for a suitable  $u \in U$ . The vector  $y = w + u = v + (u_1 + u)$  is of the form  $v + (\text{a vector of } U)$ . So  $y \in v + U = P$ . This shows that  $w + U \subset P$ . Hence,  $w + U = P$ .

(b) Let  $v_1, v_2$  be in the same parallel of  $U$ , namely,  $v + U$ . So  $v_1 = v + u_1$  for some  $u_1 \in U$ , and  $v_2 = v + u_2$  for some  $u_2 \in U$ . Then  $v_1 - v_2 = (v + u_1) - (v + u_2) = u_1 - u_2$ , which belongs to  $U$ . Conversely, if  $v_1 - v_2 \in U$ , then  $v_1 - v_2 = u$  for some  $u \in U$ . So  $v_1 = v_2 + u$ . Therefore,  $v_1 \in v_2 + U$ . Also  $v_2 = v_2 + 0$ . Therefore,  $v_2 \in v_2 + U$ , since  $0 \in U$ . So  $v_1$  and  $v_2$  belong to the same parallel  $v_2 + U$ . ■

As illustration, take  $V = V_3$  and  $U = yz$ -plane. Let  $v = (1, 1, 1)$ . Then  $P = v + U$  is the parallel given by the set

$$\begin{aligned} & \{(1, 1, 1) + (0, \beta, \gamma) \mid \beta, \gamma \text{ arbitrary scalars}\} \\ &= \{(1, 1 + \beta, 1 + \gamma) \mid \beta, \gamma \text{ arbitrary scalars}\}. \end{aligned}$$

Part (a) of Theorem 3.4.6 says that to describe this set we could take instead of  $(1, 1, 1)$ , any other vector from  $P$ . Let us take vector  $(1, 0, 0)$ , which is also in  $P$ . The theorem says that every vector

$$(1, 1, 1) + (0, \beta, \gamma)$$

can also be written in the form

$$(1, 0, 0) + (0, \beta', \gamma')$$

for suitable  $\beta'$  and  $\gamma'$ . Clearly,  $\beta' = 1 + \beta$  and  $\gamma' = 1 + \gamma$ .

To continue the illustration, both  $(1, 1, 1)$  and  $(1, 0, 0)$  are in  $P$ . The difference  $(0, 1, 1)$  is obviously in  $U$ . Part (b) of the theorem says that whenever the difference of two vectors belongs to  $U$ , then they both belong to the same parallel and conversely.

**3.4.7 Remark** Starting from a subspace  $U$ , we can obtain many parallels of  $U$ . This is the same as starting from one straight line through the origin and drawing many parallel straight lines through different points in the plane. Given a vector  $v \in V$ , we get a parallel  $v + U$ . If we take another vector  $v' \in V$ , the parallel  $v' + U$  obtained from  $v'$  will be different from  $v + U$  iff  $v'$  is not in  $v + U$ .

### Problem Set 3.4

- Let  $U_1, U_2, \dots, U_n$  be  $n$  subspaces of a vector space  $V$ . Then prove that  $U_1 \cap U_2 \cap \dots \cap U_n = \bigcap_{i=1}^n U_i$  is also a subspace of  $V$ .
- If  $U$  and  $W$  are subspaces of a vector space  $V$ , prove that
  - $U \cap W$  is a subspace of  $W$
  - $U \cup W$  is a subspace of  $V$  iff  $U \subset W$  or  $W \subset U$ .
- If  $S$  is a nonempty subset of  $V$ , prove that  $[S]$  is the intersection of all subspaces of  $V$  containing  $S$ .
- Prove that the set of vectors  $(x_1, x_2, \dots, x_n) \in V_n$  satisfying the following  $m$  conditions is a subspace of  $V_n$ :

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1n}x_n &= 0 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \dots + \alpha_{2n}x_n &= 0 \\ \vdots &\vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \dots + \alpha_{mn}x_n &= 0, \end{aligned}$$

$\alpha_{ij}$ 's are fixed scalars.

- Find the intersection of the given sets  $U$  and  $W$  and determine whether it is a subspace.
  - $U = \{(x_1, x_2) \in V_2 \mid x_1 \geq 0\}$ ,  $W = \{(x_1, x_2) \in V_2 \mid x_1 < 0\}$
  - $U = \{f \in \mathcal{C}(-2, 2) \mid f(-1) = 0\}$ ,  
 $W = \{f \in \mathcal{C}(-2, 2) \mid f(1) = 0\}$

$$(c) \quad U = \{f \in \mathcal{C}(-2, 2) \mid \lim_{x \rightarrow 1} f(x) = 0\},$$

$$W = \{f \in \mathcal{C}(-2, 2) \mid \lim_{x \rightarrow 2} f(x) = 1\}$$

$$(d) \quad U = \mathcal{P}, W = \{f \in \mathcal{C}(-\infty, \infty) \mid f(x) = f(-x)\}.$$

6. Describe  $A + B$  for the given subsets  $A$  and  $B$  of  $V_2$  and determine in each case whether it is a subspace or a parallel or just a subset of  $V_2$ .

$$(a) \quad A = \{(1, 2), (0, 1)\}, B = \{(1, 0), (3, -1)\}$$

$$(b) \quad A = \{(1, -2), (5, 1)\}, B = \{(3, 5), (\frac{1}{2}, 3), (\sqrt{2}, \pi)\}$$

$$(c) \quad A = \{(\frac{1}{2}, \frac{3}{4})\}, B = \text{segment joining } (1, 1) \text{ and } (2, 3)$$

$$(d) \quad A = \{(2, 3)\}, B = \{t(3, 4) \mid 1 < t < 2\}$$

$$(e) \quad A = \{(3, 7)\}, B = \{t(-1, 2) \mid 0 < t < 1\}$$

$$(f) \quad A = \{(\frac{1}{2}, 2)\}, B = \{t(3, 0) \mid t \geq 0\}$$

$$(g) \quad A = \{(2, 4)\}, B = \{(x, y) \mid 2x + 3y = 1\}$$

$$(h) \quad A = \{(1, 5)\}, B = \{(x, y) \mid x^2 + y^2 = 1\}$$

$$(i) \quad A = \{t(3, 4) \mid 0 < t < 1\}, B = \{t(2, 5) \mid 1 < t < 2\}$$

$$(j) \quad A = \{t(1, 0) \mid 0 < t < 1\}, B = \{t(0, 1) \mid 2 < t < 4\}$$

$$(k) \quad A = \{t(1, 0) \mid t \text{ a scalar}\}, B = [(1, 2)]$$

$$(l) \quad A = \text{line } x = 3t, y = 4t, B = \text{line } 2x + 5y = 0.$$

7. Describe  $A + B$  for the given subsets  $A$  and  $B$  of  $V_3$ . Determine in each case whether  $A + B$  is a subspace or a parallel or just a subset of  $V_3$ .

$$(a) \quad A = \{(1, 2, 1)\}, B = \{t(1, 2, 0) \mid t \text{ a scalar}\}$$

$$(b) \quad A = \{(3, 1, -1)\}, B = \{(x, y, z) \mid x + y + z = 0\}$$

$$(c) \quad A = \{(1, -3, 4)\}, B = [(1, 2, 3), (0, 0, 1)]$$

$$(d) \quad A = [(1, 2, 3)], B = [(3, 1, 0)]$$

$$(e) \quad A = [(1/2, 2/3, 1)], B = \text{plane } 2x + 3y + z = 0$$

$$(f) \quad A = [(5, 0, 2)], B = [(1, 2, 3), (0, 1, 2)]$$

$$(g) \quad A = [(1, 0, -1)], B = [(2, 5, 8), (2, 3, 4)].$$

8. If  $U$  and  $W$  are two subspaces of a vector space  $V$ , prove that  $U + W = U \text{ iff } W \subset U$ .

9. Let  $A$  and  $B$  be two nonempty finite subsets of a vector space  $V$ . Then prove that

$$(a) \quad [A \cap B] \subset [A] \cap [B]$$

$$(b) \quad [A \cup B] = [A] + [B].$$

10. If  $U_1, U_2$ , and  $U_3$  are three subspaces of a vector space  $V$ , prove that

$$(U_1 \cap U_2) + (U_2 \cap U_3) \subset (U_1 + U_2) \cap U_3.$$

11. (a) Prove that two parallel straight lines in  $V_2$  are parallels of the same nontrivial subspace of  $V_2$ .

- (b) Prove that the sum of two distinct intersecting lines in  $V_2$  is  $V_2$ . Is it a direct sum?

12. (a) Show that the set of all solutions of the equation  $3x_1 + 2x_2 - x_3 + x_4 = 5$  can be expressed as  $(1, -2, 2, 8) + W$ , where  $W$  is the space of all solutions of the equation  $3x_1 + 2x_2 - x_3 + x_4 = 0$ .
- (b) Let  $W$  be the set of all solutions of the equation  $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \beta$ . Then show that  $W$  is a parallel in  $V_n$ . Find its base space and a leader.
13. Show that the given set  $P$  is a parallel in  $(\mathcal{C}(0, 2))$ . In each case find the base space and a leader.
- (a)  $P = \{f \mid f'(x) = 3x^2\}$
- (b)  $P = \{f \mid f'(x) = xe^{x^2}\}$
- (c)  $P = \{f \mid f''(x) = \sin x\}$
- (d)  $P = \{f \mid f''(x) = x^2 + 2x + 3\}$
- (e)  $P = \{f \mid f(1) = 2\}$ .
14. Let  $U = \{f \in \mathcal{C}'(-a, a) \mid f \text{ is odd}\}$ , and  $W = \{f \in \mathcal{C}(-a, a) \mid f \text{ is even}\}$ . Then describe (a)  $U \cap W$  and (b)  $U + W$ , and determine whether they are subspaces of  $\mathcal{C}'(-a, a)$ . Is  $\mathcal{C}'(-a, a) \subset U + W$ ? Is  $U + W$  a direct sum? Is  $\mathcal{C}'(-a, a) = U \oplus W$ ?
15. Prove that a linear variety is a subspace iff its leader belongs to the base space.
16. True or false?
- (a) For any subset  $A$  of a vector space  $V$ ,  $[A] = [A + A]$ .
- (b) If  $U_i, i = 1, 2, \dots$  are subspaces of  $V$ , then  $\bigcap_{i=1}^{\infty} U_i$  is a subspace of  $V$ .
- (c) The intersection of two linear varieties is a linear variety, provided it is nonempty.
- (d) The sum of two linear varieties may not be a linear variety.
- (e) The union of two linear varieties need not be a linear variety.
- (f) The set of all solutions of the equation  $\frac{d^2 y}{dt^2} = g$ , where  $g$  is a constant, is a linear variety in  $\mathcal{C}(0, \infty)$ .
- (g) In Example 3.14  $V_n$  cannot be replaced by  $V_n^C$ .

### 3.5 LINEAR DEPENDENCE, INDEPENDENCE

We start this article with the definition of 'trivial' and 'nontrivial' linear combinations.



**3.5 1 Definition** If  $u_1, u_2, \dots, u_n$  are  $n$  vectors of a vector space  $V$ , then the linear combination

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n \quad (1)$$

is called a *trivial linear combination* if all the scalars  $\alpha_1, \alpha_2, \dots, \alpha_n$  are zero. Otherwise the linear combination is said to be *nontrivial*.

In other words, a nontrivial linear combination is of the form (1), where at least one of the  $\alpha$ 's is not zero. As illustration,

$0u_1 + 0u_2 + \dots + 0u_n$  is a trivial linear combination;

$0u_1 + cu_2 + \dots + 0u_{n-1} + 1u_n$  is a nontrivial linear combination;

and

$1u_1 + 2u_2 + 3u_3 + \dots + nu_n$  is also a nontrivial linear combination.

Note that the trivial linear combination of any set of vectors is always the zero vector, for

$$0u_1 + 0u_2 + \dots + 0u_n = 0 + 0 + \dots + 0 = 0.$$

The question arises whether a nontrivial linear combination of a set of vectors can give the zero vector. The answer is in the affirmative. Consider the following examples.

**Example 3.20** Let  $(1, 0, 0)$ ,  $(2, 0, 0)$ , and  $(0, 0, 1)$  be three vectors in  $V_3$ . Then we have

$$1(1, 0, 0) + (-\frac{1}{2})(2, 0, 0) + 0(0, 0, 1) = (0, 0, 0) = 0.$$

Thus, a nontrivial linear combination may give the zero vector.

**Example 3.21** Let  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$  be three vectors of  $V_3$ . Take any linear combination of these vectors, say

$$\alpha(1, 0, 0) + \beta(0, 1, 0) + \gamma(0, 0, 1),$$

which is the same as  $(\alpha, \beta, \gamma)$ . If this were to be the zero vector, then  $(\alpha, \beta, \gamma) = (0, 0, 0)$ . Therefore,  $\alpha = 0 = \beta = \gamma$ .

In this case the only linear combination that gives the zero vector is the trivial linear combination. The same is true in the following example.

**Example 3.22** Let  $(1, 1, 1)$ ,  $(1, 1, 0)$ , and  $(1, 0, 1)$  be three vectors. As before, if

$$\alpha(1, 1, 1) + \beta(1, 1, 0) + \gamma(1, 0, 1) = (0, 0, 0),$$

then

$$(\alpha + \beta + \gamma, \alpha + \beta, \alpha + \gamma) = (0, 0, 0).$$

Therefore,  $\alpha + \beta + \gamma = 0$ ,  $\alpha + \beta = 0$ , and  $\alpha + \gamma = 0$ . These give  $\alpha = 0 = \beta = \gamma$ .

It is clear that these three examples fall into two categories. Examples 3.21 and 3.22 are of one kind, whereas Example 3.20 is of another. In Example 3.20,  $(1, 0, 0)$  is a linear combination of  $(2, 0, 0)$  and  $(0, 0, 1)$ :

$$(1, 0, 0) = \frac{1}{2}(2, 0, 0) + 0(0, 0, 1).$$

In other words,  $(1, 0, 0)$  depends on  $(2, 0, 0)$  and  $(0, 0, 1)$ . We say that these three vectors are linearly dependent. On the other hand, in Examples

3.21 and 3.22 there is no such dependence of one vector on the others. We say that they are linearly independent.

We formalise these concepts of linear dependence and linear independence in the following definitions.

**3.5.2 Definition** A set  $\{u_1, u_2, \dots, u_n\}$  of vectors is said to be *linearly dependent* (LD) if there exists a nontrivial linear combination of  $u_1, u_2, \dots, u_n$  that equals the zero vector.

**Example 3.23** Prove that the vectors  $(1, 0, 1)$ ,  $(1, 1, 0)$ , and  $(-1, 0, -1)$  are LD.

To prove linear dependence we must try to find scalars  $\alpha, \beta, \gamma$  such that

$$\alpha(1, 0, 1) + \beta(1, 1, 0) + \gamma(-1, 0, -1) = 0 = (0, 0, 0).$$

Therefore,  $\alpha + \beta - \gamma = 0$ ,  $\beta = 0$ , and  $\alpha - \gamma = 0$ . This can happen iff  $\alpha = \gamma$ . Any nonzero value for  $\alpha$ , say 1, will do. Thus

$$1(1, 0, 1) + 0(1, 1, 0) + 1(-1, 0, -1) = 0.$$

So there exists a nontrivial linear combination of the given vectors, which equals the zero vector. Hence, the vectors are LD.

Once we find a nontrivial linear combination equal to the zero vector, linear dependence is proved. Very often it is possible to guess such a linear combination.

**3.5.3 Definition** A set  $\{u_1, u_2, \dots, u_n\}$  of vectors is said to be *linearly independent* (LI) if no nontrivial linear combination of  $u_1, u_2, \dots, u_n$  equals the zero vector.

'No nontrivial linear combination equals the zero vector' means the following :

If at all there is a linear combination that equals the zero vector, then it must be the trivial linear combination. Now recall that the trivial linear combination is always zero. So the statement within quotation marks means 'The *only* linear combination that equals the zero vector is the trivial linear combination'. In view of this, Definition 3.5.3 can be reworded as follows.

**3.5.4 Definition (Reworded)** A set  $\{u_1, u_2, \dots, u_n\}$  of vectors is said to be *linearly independent* (LI) if the *only* linear combination of  $u_1, u_2, \dots, u_n$  that equals the zero vector is the trivial linear combination.

By convention, the empty set is considered to be LI. Note that linear dependence and linear independence are opposite concepts.

**Example 3.24** Prove that the vectors  $(1, 0, 1)$ ,  $(1, 1, 0)$ , and  $(1, 1, -1)$  are LI.

If possible, let

$$\alpha(1, 0, 1) + \beta(1, 1, 0) + \gamma(1, 1, -1) = 0 = (0, 0, 0). \quad (2)$$

This gives  $\alpha + \beta + \gamma = 0$ ,  $\beta + \gamma = 0$ , and  $\alpha - \gamma = 0$ . Therefore,  $\alpha = 0$

$= \beta = \gamma$ . This means that the only values of  $\alpha, \beta, \gamma$  that satisfy Equation (2) are  $\alpha = 0 = \beta = \gamma$ , i.e. only the trivial linear combination equals the zero vector. Hence, the given vectors are LI.

## HOW TO CHECK LINEAR DEPENDENCE OR INDEPENDENCE

The foregoing discussion tells us that when we are given a set  $\{u_1, u_2, \dots, u_n\}$  of vectors and we want to check whether it is LD or LI, the following procedure will be the natural one.

First, we assume that some linear combination of  $u_1, u_2, \dots, u_n$  is equal to the zero vector, that is,

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = \mathbf{0}. \quad (3)$$

Then two cases arise. One is when it is possible to *find* scalars  $\alpha_1, \alpha_2, \dots, \alpha_n$  with at least one of them not zero such that Equation (3) is satisfied. In this case we conclude that the vectors  $u_1, u_2, \dots, u_n$  are LD.

The other case is when we can *prove* that our assumption automatically implies that  $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_n$ . In other words, the only values of  $\alpha_i$ 's that satisfy Equation (3) are  $\alpha_i = 0$  for each  $i = 1, 2, \dots, n$ . In this case we conclude that  $u_1, u_2, \dots, u_n$  are LI.

In the first case, even guesswork in finding the suitable scalars is enough, whereas in the second case a proof is necessary.

**Example 3.25** Check whether the following set of vectors is LD or LI :

$$\{(1, 0, 1), (1, 1, 0), (1, -1, 1), (1, 2, -3)\}.$$

Suppose

$$\alpha(1, 0, 1) + \beta(1, 1, 0) + \gamma(1, -1, 1) + \delta(1, 2, -3) = \mathbf{0} = (0, 0, 0)$$

$$\text{or } (\alpha + \beta + \gamma + \delta, \beta - \gamma + 2\delta, \alpha + \gamma - 3\delta) = (0, 0, 0)$$

$$\text{or } \alpha + \beta + \gamma + \delta = 0, \beta - \gamma + 2\delta = 0, \alpha + \gamma - 3\delta = 0.$$

This system has a solution  $\alpha = 5\delta, \beta = -4\delta, \gamma = -2\delta$  for each choice of  $\delta$ . Take  $\delta = 1$ . This gives

$$5(1, 0, 1) - 4(1, 1, 0) - 2(1, -1, 1) + 1(1, 2, -3) = (0, 0, 0).$$

Hence, the given set is LD.

**Example 3.26** Check the linear dependence or linear independence of the set  $\{e^x, e^{2x}\}$  in  $\mathcal{C}^{(\infty)}(-\infty, \infty)$ .

Suppose that

$$\alpha e^x + \beta e^{2x} = \mathbf{0} \quad (x) = 0 \text{ for all } x \in (-\infty, \infty).$$

This gives, on differentiation,

$$\alpha e^x + 2\beta e^{2x} = 0 \text{ for all } x \in (-\infty, \infty).$$

Solving these two equations for  $\alpha$  and  $\beta$ , we get

$$\beta e^{2x} = 0 \text{ for all } x \in (-\infty, \infty);$$

but  $e^{2x}$  is never zero and hence  $\beta = 0$ . It then automatically follows that  $\alpha$  is also zero. Thus, the only linear combination that equals the zero

vector is the trivial one. Hence, the given set is LI in  $\mathcal{C}^{(\infty)}(-\infty, \infty)$ . We also say that the set is LI over  $(-\infty, \infty)$ .

**Example 3.27** Check the linear dependence or linear independence of the set  $\{x, |x|\}$  in  $\mathcal{C}(-1, 1)$ .

Suppose  $\alpha x + \beta |x| = 0$ .

Since the function  $|x|$  is not differentiable at zero, we cannot use the method in Example 3.26.

$\alpha x + \beta |x| = 0$  holds for all  $x$  in  $(-1, 1)$ . So choosing two different values of  $x$ , say  $x = 1/2$  and  $x = -1/2$ , we get

$$\frac{\alpha}{2} + \frac{\beta}{2} = 0 \text{ and } -\frac{\alpha}{2} + \frac{\beta}{2} = 0.$$

Solving these two equations, we get  $\alpha = 0 = \beta$ . Hence, the set is LI over  $(-1, 1)$ .

We shall now take up the geometrical meaning of linear dependence. For this we need the following definitions in a vector space  $V$ .

**3.5.5 Definition** Given a vector  $v \neq 0$ , the set of all scalar multiples of  $v$  is called *the line through  $v$* .

Geometrically, in the cases of  $V_1$ ,  $V_2$ , and  $V_3$  it is nothing but the straight line through the origin and  $v$ .

**3.5.6 Definition** Two vectors  $v_1$  and  $v_2$  are *collinear* if one of them lies in the 'line' through the other.

Clearly,  $0$  is collinear with any nonzero vector  $v$ .

**3.5.7 Definition** Given two vectors  $v_1$  and  $v_2$ , which are not collinear, their span, namely,  $[v_1, v_2]$ , is called *the plane through  $v_1$  and  $v_2$* .

Geometrically, in the cases of  $V_2$  and  $V_3$  it is nothing but the plane passing through the origin,  $v_1$  and  $v_2$ .

**3.5.8 Definition** Three vectors  $v_1$ ,  $v_2$ , and  $v_3$  are *coplanar* if one of them lies in the 'plane' through the other two.

Clearly,  $0$  is coplanar with every pair of noncollinear vectors.

**Example 3.28** The vectors  $v$  and  $2v$  of a vector space  $V$  are collinear, because  $2v$  lies in the 'line' through  $v$ , i.e.  $2v$  is a scalar multiple of  $v$ . In particular,  $\sin x$  and  $2 \sin x$  are collinear in  $\mathcal{F}(I)$ .

**Example 3.29** The functions  $\sin x$  and  $\cos x$  in  $\mathcal{F}(I)$  are not collinear, because neither of the two lies in the 'line' through the other, i.e. one is not a scalar multiple of the other. Their span, namely,

$$[\sin x, \cos x] = \{\alpha \sin x + \beta \cos x \mid \alpha, \beta \text{ any scalars}\}$$

is the plane through the vectors  $\sin x$  and  $\cos x$ .

**Example 3.30** The functions  $\sin x$ ,  $\cos x$ ,  $\tan x$  in  $\mathcal{F}(I)$  are obviously not coplanar, because none of them lies in the 'plane' through the other two.

**Example 3.31** The functions  $\sin^2 x$ ,  $\cos^2 x$ ,  $\cos 2x$  are coplanar.  $\cos 2x$  lies in the plane through  $\sin^2 x$  and  $\cos^2 x$ , since  $\cos 2x$  is a linear combination of  $\cos^2 x$  and  $\sin^2 x$ .

**3.5.9 Theorem** Let  $V$  be any vector space. Then

- (a) The set  $\{v\}$  is LD iff  $v = 0$ .
- (b) The set  $\{v_1, v_2\}$  is LD iff  $v_1$  and  $v_2$  are collinear, i.e. one of them is a scalar multiple of the other.
- (c) The set  $\{v_1, v_2, v_3\}$  is ED iff  $v_1, v_2$ , and  $v_3$  are coplanar, i.e. one of them is a linear combination of the other two.

*Proof:* (a)  $\{v\}$  is LD iff there exists a nonzero scalar  $\alpha$  such that  $\alpha v = 0$ . This is possible (Problem 5, Problem Set 3.1) iff  $v = 0$ .

(b) Suppose  $\{v_1, v_2\}$  is LD. Then there exist scalars  $\alpha, \beta$  with at least one of them, say  $\alpha \neq 0$ , such that  $\alpha v_1 + \beta v_2 = 0$ . Therefore,  $v_1 = (-\beta/\alpha) v_2$ , which means  $v_1$  is a scalar multiple of  $v_2$ , i.e.  $v_1$  lies in the line through  $v_2$ . Hence,  $v_1, v_2$  are collinear.

Conversely, if  $v_1$  and  $v_2$  are collinear, then, by definition, one of them, say  $v_1$ , lies in the line through  $v_2$ . Therefore,  $v_1$  is a scalar multiple of  $v_2$ . So  $v_1 = \alpha v_2$ , i.e.  $1 \cdot v_1 - \alpha v_2 = 0$ . Hence,  $v_1$  and  $v_2$  are LD.

(c) Suppose  $\{v_1, v_2, v_3\}$  is LD. Then there exist scalars  $\alpha, \beta$ , and  $\gamma$  with at least one of them, say  $\alpha \neq 0$ , such that

$$\alpha v_1 + \beta v_2 + \gamma v_3 = 0$$

or

$$v_1 = (-\beta/\alpha) v_2 + (-\gamma/\alpha) v_3.$$

This means  $v_1 \in [v_2, v_3]$ . Hence,  $v_1$  lies in the plane through  $v_2$  and  $v_3$ . So  $v_1, v_2$ , and  $v_3$  are coplanar.

Conversely, if  $v_1, v_2$ , and  $v_3$  are coplanar, then one of them, say  $v_1 \in [v_2, v_3]$ , i.e.  $v_1 = \alpha_2 v_2 + \alpha_3 v_3$  for suitable scalars  $\alpha_2$  and  $\alpha_3$ . Therefore,  $1v_1 - \alpha_2 v_2 - \alpha_3 v_3 = 0$ . Hence,  $v_1, v_2$ , and  $v_3$  are LD.

As illustration, consider the three vectors  $(1, 1, 1)$ ,  $(1, -1, 1)$ , and  $(3, -1, 3)$ . They are LD, for

$$1(1, 1, 1) + 2(1, -1, 1) - 1(3, -1, 3) = 0.$$

Hence, by Theorem 3.5.9, the plane through  $(1, 1, 1)$  and  $(3, -1, 3)$  contains the point  $(1, -1, 1)$ . Let us verify this.

The plane through  $(1, 1, 1)$  and  $(3, -1, 3)$  is

$$[(1, 1, 1), (3, -1, 3)] = \{(\alpha + 3\beta, \alpha - \beta, \alpha + 3\beta) \mid \alpha, \beta \text{ any scalars}\}.$$

This set contains  $(1, -1, 1)$ , for we have only to choose  $\alpha$  and  $\beta$  such that

$$\alpha + 3\beta = 1, \alpha - \beta = -1, \alpha + 3\beta = 1.$$

This gives  $\alpha = -1/2$  and  $\beta = 1/2$ .

Before we proceed, we shall record some simple facts about linear dependence.

**3.5.10 Fact** In a vector space  $V$  any set of vectors containing the zero vector is LD.

For, if  $\{v_1, v_2, \dots, v_n\}$  is a set and  $v_i = 0$ ,  
then

$$0v_1 + 0v_2 + \dots + 0v_{i-1} + 1v_i + 0v_{i+1} + \dots + 0v_n$$

is a nontrivial linear combination resulting in the zero vector.

**3.5.11 Fact** In a vector space  $V$ , if  $v$  is a linear combination of  $v_1, v_2, \dots, v_n$ , i.e.  $v \in [v_1, v_2, \dots, v_n]$ , then  $\{v, v_1, \dots, v_n\}$  is LD.

For,  $v \in [v_1, v_2, \dots, v_n]$  means

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n,$$

i.e.  $1v - \alpha_1 v_1 - \dots - \alpha_n v_n = 0$ .

**3.5.12 Fact** In a vector space  $V$ , if the set  $\{v_1, v_2, \dots, v_n\}$  is LI and  $v \notin [v_1, v_2, \dots, v_n]$ , then the set  $\{v, v_1, v_2, \dots, v_n\}$  is LI.

The proof is left to the reader.

In the case where the set consists of just two or three vectors, we can look at these facts from the 'geometric' point of view of Theorem 3.5.9. For the purpose of understanding, imagine that  $V = V_3$  (though this is not necessary for the argument). Since  $0$  lies in the line through  $v$ ,  $\{0, v\}$  is always LD by Theorem 3.5.9 (b). Similarly,  $0$  always lies in the plane through any two vectors  $v_1, v_2$ . Hence,  $\{0, v_1, v_2\}$  is always LD by Theorem 3.5.9 (c).

The idea contained in Theorem 3.5.9 does not stop with just three vectors. We can in fact work out a general result on the same lines for a linearly dependent set of  $n$  vectors. Before we do that, let us look at the theorem once again. Actually, in the proof, we have indicated the following :

(i)  $\{v_1, v_2\}$  is LD iff one of the vectors is a scalar multiple of the other and (ii)  $\{v_1, v_2, v_3\}$  is LD iff one of them is a linear combination of the others.

In general, we shall now prove in Theorem 3.5.14 that if  $\{v_1, v_2, \dots, v_n\}$  is LD, then one of the vectors is a linear combination of the others, the converse of which is Fact 3.5.11. Before that, we state the following simple theorem and leave its proof to the reader.

**3.5.13 Theorem** (a) If a set is LI, then any subset of it is also LI, and  
(b) If a set is LD, then any superset of it is also LD.

**3.5.14 Theorem** In a vector space  $V$  suppose  $\{v_1, v_2, \dots, v_n\}$  is an ordered set of vectors with  $v_1 \neq 0$ . The set is LD iff one of the vectors  $v_2, v_3, \dots, v_n$ , say  $v_k$ , belongs to the span of  $v_1, \dots, v_{k-1}$ , i.e.  $v_k \in [v_1, v_2, \dots, v_{k-1}]$  for some  $k = 2, 3, \dots, n$ .

*Proof:* Suppose  $v_k \in [v_1, v_2, \dots, v_{k-1}]$ . Then  $v_k$  is a linear combination of  $v_1, v_2, \dots, v_{k-1}$ . Thus, the set  $\{v_1, v_2, \dots, v_{k-1}, v_k\}$  is LD. Hence, by Theorem 3.5.13 (b),  $\{v_1, v_2, \dots, v_n\}$  is LD

Conversely, suppose that  $\{v_1, v_2, \dots, v_n\}$  is LD. Consider the sets

$$\begin{aligned} S_1 &= \{v_1\} \\ S_2 &= \{v_1, v_2\} \\ &\vdots \\ S_i &= \{v_1, v_2, \dots, v_i\} \\ &\vdots \\ S_n &= \{v_1, v_2, \dots, v_n\}. \end{aligned}$$

By Theorem 3.5.9 (a),  $S_1$  is LI and by assumption  $S_n$  is LD. So we go down the list and choose the first linearly dependent set. Let  $S_k$  be that set. Then  $S_k$  is LD and  $S_{k-1}$  is LI. Here  $2 \leq k \leq n$ .

Since  $S_k$  is LD, we have

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_k v_k = 0,$$

with at least one of the  $\alpha$ 's not zero. Surely  $\alpha_k \neq 0$ , for if  $\alpha_k = 0$ , then  $S_{k-1}$  would become a linearly dependent set contradicting our statement that  $S_k$  is the first linearly dependent set. Therefore, we can write

$$v_k = \left(-\frac{\alpha_1}{\alpha_k}\right) v_1 + \left(-\frac{\alpha_2}{\alpha_k}\right) v_2 + \dots + \left(-\frac{\alpha_{k-1}}{\alpha_k}\right) v_{k-1},$$

which means  $v_k \in [v_1, v_2, \dots, v_{k-1}]$ .

**3.5.15 Corollary** *A finite subset  $S = \{v_1, v_2, \dots, v_n\}$  of a vector space  $V$  containing a nonzero vector has a linearly independent subset  $A$  such that  $[A] = [S]$ .*

*Proof:* We assume  $v_1 \neq 0$ . If  $S$  is LI, then there is nothing to prove, as we have  $A = S$ . If not, then, by Theorem 3.5.14, we have a vector  $v_k$  such that  $v_k \in [v_1, \dots, v_{k-1}]$ . Discard  $v_k$ . The remaining set  $S_1 = \{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_n\}$  has the same span as that of  $S$  (see Problem 4 (c), Problem Set 3.3). If  $S_1$  is LI, then we are done. If not, then repeat the foregoing process. This must finally lead to a linearly independent subset  $A$  such that  $[A] = [S]$ . (Why?)

**Example 3.32** Show that the ordered set  $\{(1, 1, 0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}$  is LD and locate one of the vectors that belongs to the span of the previous ones.

Consider the sets

$$\begin{aligned} S_1 &= \{(1, 1, 0)\} \\ S_2 &= \{(1, 1, 0), (0, 1, 1)\} \\ S_3 &= \{(1, 1, 0), (0, 1, 1), (1, 0, -1)\} \\ S_4 &= \{(1, 1, 0), (0, 1, 1), (1, 0, -1), (1, 1, 1)\}. \end{aligned}$$

Obviously,  $S_1$  is LI.  $S_2$  is also LI, because neither of the two vectors in  $S_2$  is a scalar multiple of the other.  $S_3$  is LD, because

$$1(1, 1, 0) - 1(0, 1, 1) - 1(1, 0, -1) = (0, 0, 0).$$

Hence,  $(1, 0, -1) \in [(1, 1, 0), (0, 1, 1)]$ .  $S_4$  is LD, because of Theorem 3.5.13 (b).

**Example 3.33** In Example 3.32 find the largest linearly independent subset whose span is  $[S_4]$ .

As proved in Example 3.32,  $(1, 0, -1) \in [(1, 1, 0), (0, 1, 1)]$  and hence  $(1, 0, -1) \in [(1, 1, 0), (0, 1, 1), (1, 1, 1)]$ . Therefore, we discard  $(1, 0, -1)$ . The span of the remaining set  $A = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$  is the same as  $[S_4]$ . Now we check for the linear independence of  $A$ . Suppose

$$\alpha(1, 1, 0) + \beta(0, 1, 1) + \gamma(1, 1, 1) = (0, 0, 0)$$

$$\text{or} \quad (\alpha + \gamma, \alpha + \beta + \gamma, \beta + \gamma) = (0, 0, 0).$$

This means  $\alpha + \gamma = 0$ ,  $\alpha + \beta + \gamma = 0$ ,  $\beta + \gamma = 0$ . Solving these equations, we get only one solution, i.e.  $\alpha = 0 = \beta = \gamma$ . Hence, the set is LI. Therefore,  $A = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$  is the largest linearly independent subset and  $[A] = [S_4]$ .

Finally, let us extend the concept of linear dependence and linear independence to infinite sets.

**3.5.16 Definition** An infinite subset  $S$  of a vector space  $V$  is said to be *linearly independent* (LI) if every finite subset of  $S$  is LI.

$S$  is said to be *linearly dependent* (LD), if it is not LI.

**Example 3.34** The subset

$$S = \{1, x, x^2, x^3, \dots\}$$

of  $\mathcal{P}$  is LI.

For, suppose  $a_1x^{k_1} + a_2x^{k_2} + \dots + a_nx^{k_n} = 0$  with  $k_1, k_2, \dots, k_n$  being distinct nonnegative integers. Note that the equality is an algebraic identity, since the right-hand side is the zero polynomial. So, either by giving various values to  $x$  or by repeated differentiation of both sides of the identity, we get

$$a_1 = 0 = a_2 = \dots = a_n.$$

### Problem Set 3.5

1. Which of the following subsets  $S$  of  $V_3$  are LI ?

- (a)  $S = \{(1, 2, 1), (-1, 1, 0), (5, -1, 2)\}$
- (b)  $S = \{(1, 0, 0), (1, 1, 1), (1, 2, 3)\}$
- (c)  $S = \{(1, 1, 2), (-3, 1, 0), (1, -1, 1), (1, 2, -3)\}$
- (d)  $S = \{(1, 5, 2), (0, 0, 1), (1, 1, 0)\}$
- (e)  $S = \{(1/2, 1/3, 1), (0, 0, 0), (2, 3/4, -1/3)\}$ .

2. Which of the following subsets  $S$  of  $V_4$  are LD ?

- (a)  $S = \{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 1), (0, 0, 1, 1)\}$
- (b)  $S = \{(1, -1, 2, 0), (1, 1, 2, 0), (3, 0, 0, 1), (2, 1, -1, 0)\}$
- (c)  $S = \{(1, 1, 1, 0), (3, 2, 2, 1), (1, 1, 3, -2), (1, 2, 6, -5), (1, 1, 2, 1)\}$



- (d)  $S = \{(1/2, 2/3, 3/4, 4), (0, 0, 2, 0), (1, 2, 0, 1), (1/2, 2/3, 3/4, 4)\}$   
 (e)  $S = \{(1, 2, 3, 0), (-1, 7, 3, 3), (1, -1, 1, -1)\}$ .
3. Which of the following subsets  $S$  of  $\mathcal{P}$  are LI ?  
 (a)  $S = \{x^2 - 1, x + 1, x - 1\}$   
 (b)  $S = \{1, x + x^2, x - x^2, 3x\}$   
 (c)  $S = \{x, x^3 - x, x^4 + x^2, x + x^2 + x^4 + 1/2\}$   
 (d)  $S = \{x^2 - 4, x + 2, x - 2, x^2/3\}$ .
4. Which of the following subsets  $S$  of  $\mathcal{C}(0, \infty)$  are LI ?  
 (a)  $S = \{x, \sin x, \cos x\}$   
 (b)  $S = \{\sin^2 x, \cos 2x, 1\}$   
 (c)  $S = \{\sin x, \cos x, \sin(x + 1)\}$   
 (d)  $S = \{\ln x, \ln x^2, \ln x^3\}$   
 (e)  $S = \{x^2 e^x, x e^x, (x^2 + x - 1)e^x\}$ .
5. Prove that the vectors  $(a, b), (c, d)$  are LD iff  $ad = bc$ .
6. Prove that  
 (a) 1 and  $i$  are LD in the set  $C$  of complex numbers considered as a complex vector space  
 (b) 1 and  $i$  are LI in the set  $C$  of complex numbers considered as a real vector space.
7. Show that the set  $S = \{\sin x, \sin 2x, \dots, \sin nx\}$  is a linearly independent subset of  $\mathcal{C}[-\pi, \pi]$  for every positive integer  $n$ .
8. Prove Theorem 3.5.13.
9. In the proof of Corollary 3.5.15 answer the question 'why'.
10. If  $u, v$ , and  $w$  are three linearly independent vectors of a vector space  $V$ , then prove that  $u + v, v + w$ , and  $w + u$  are also LI.
11. Find a linearly independent subset  $A$  of  $S$  in Problem 1 such that  $[A] = [S]$ .
12. Proceed as in Problem 11 for the sets  $S$  in Problem 2.
13. Proceed as in Problem 11 for the sets  $S$  in Problem 3.
14. Proceed as in Problem 11 for the sets  $S$  in Problem 4.
15. In Problems 1, 2, 3, and 4, whenever a set is LD, locate one of the vectors that is in the span of the others.
16. True or false ?  
 (a)  $\{\sin x, \cos x, 1\}$  is LD in  $\mathcal{C}(0, 1)$ .  
 (b)  $\{i + j, i - j, 2i + 3j\}$  is LD in  $V_3$ .  
 (c) If  $u_1, u_2, u_3$  are LD, then  $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0 \Rightarrow$  one of the  $\alpha$ 's is not zero.  
 (d) Every set of three vectors in  $V_2$  is LD.  
 (e) A set of two vectors in  $V_2$  is always LI.  
 (f) If a vector  $v$  is a linear combination of  $u_1, u_2, \dots, u_n$ , then  $\{u_1, u_2, \dots, u_n\}$  is LI.

- (g) A subset of a linearly dependent set can never be LD.
- (h) Every superset of a linearly independent set is LI.
- (i)  $x$  and  $|x|$  are LI in  $\mathcal{C}[1, 4]$ .

### 3.6 DIMENSION AND BASIS

We shall begin this article with the definition of a 'basis'.

**3.6.1 Definition** A subset  $B$  of a vector space  $V$  is said to be a *basis* for  $V$  if

- (a)  $B$  is linearly independent, and
- (b)  $[B] = V$ , i.e.  $B$  generates  $V$ .

Another definition of a basis, which is used by some authors, will be introduced as an equivalent property in Theorem 3.6.8.

**Example 3.35** Take  $V = V_3$ . Consider the set  $B = \{i, j, k\}$ , where  $i = (1, 0, 0)$ ,  $j = (0, 1, 0)$ ,  $k = (0, 0, 1)$ .

The set  $B$  is LI, because

$$\alpha i + \beta j + \gamma k = \alpha(1, 0, 0) + \beta(0, 1, 0) + \gamma(0, 0, 1) = \mathbf{0} = (0, 0, 0)$$

implies that  $(\alpha, \beta, \gamma) = (0, 0, 0)$ , i.e.  $\alpha = 0 = \beta = \gamma$ . Also  $B$  spans  $V_3$ , because any vector  $(x_1, x_2, x_3)$  of  $V_3$  can be written as a linear combination of  $i, j$ , and  $k$ , namely,  $(x_1, x_2, x_3) = x_1 i + x_2 j + x_3 k$ . Hence,  $B$  is a basis for  $V_3$ .

It may be seen that the set  $B_1 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$  is also a basis for  $V_3$ , because any vector  $(x_1, x_2, x_3) \in V_3$  can be written as

$$\begin{aligned} (x_1, x_2, x_3) &= \frac{x_1 + x_2 - x_3}{2} (1, 1, 0) + \frac{x_1 + x_3 - x_2}{2} (1, 0, 1) \\ &\quad + \frac{x_2 + x_3 - x_1}{2} (0, 1, 1) \end{aligned}$$

$$\text{and } \alpha(1, 1, 0) + \beta(1, 0, 1) + \gamma(0, 1, 1) = (0, 0, 0)$$

implies  $\alpha = 0 = \beta = \gamma$ , i.e.  $B$  is LI.

This example shows that a basis for a vector space  $V$  need not be unique.

The two properties of a basis  $B$ , namely, (i)  $B$  is LI and (ii)  $B$  generates  $V$  are not entirely unrelated especially when  $B$  is finite. For, if a set  $B$  of  $n$  elements generates  $V$ , then we can prove, with some effort, that no linearly independent set can have more than  $n$  vectors. This result is the content of the following important theorem.

**3.6.2 Theorem** In a vector space  $V$  if  $\{v_1, v_2, \dots, v_n\}$  generates  $V$  and if  $\{w_1, w_2, \dots, w_m\}$  is LI, then  $m \leq n$ .

In other words, we cannot have more linearly independent vectors than the number of elements in a set of generators.

**Proof:** Let us adopt the following notations. If  $S$  is a finite linearly dependent ordered set of vectors  $u_1, u_2, \dots, u_p$  with  $u_1 \neq 0$ , then

(i) by Theorem 3.5.14, there is a vector  $u_i$ ,  $2 \leq i \leq p$ , which is a linear combination of its predecessors. Let  $S'$  denote the ordered set which remains after the deletion of such a  $u_i$  and

(ii) for every vector  $w$ , let  $wS$  denote the ordered set  $\{w, u_1, u_2, \dots, u_p\}$ .

Now construct the set

$$S_1 = \{w_m, v_1, v_2, \dots, v_n\}.$$

$S_1$  has the following properties :

- (i)  $[S_1] = V$ , because  $\{v_1, \dots, v_n\}$  spans  $V$  and  $w_m \in V$ ,
- (ii)  $S_1$  is LD, since  $w_m \in V = [v_1, v_2, \dots, v_n]$ , and
- (iii)  $w_m \neq 0$ .

So, applying Theorem 3.5.14, we see that  $S'_1$  can be formed by deleting vector  $v_i$ , which is a linear combination of  $w_m, v_1, v_2, \dots, v_{i-1}$ .

Now consider the set  $S_2 = w_{m-1} S'_1$ . Since  $S'_1$  spans  $V$ ,  $S_2$  also does so. Further,  $S_2$  is LD, since  $w_{m-1} \in V = [S'_1]$  and  $w_{m-1} \neq 0$ . Therefore, by another application of Theorem 3.5.14, we form  $S'_2$ . Then we construct the set  $S_3 = w_{m-2} S'_2$  and continue the process of constructing new sets  $S$  and  $S'$ . Since the set of  $w$ 's is LI, every time the discarded element must be a  $v$ .

If all the  $w$ 's are used up in this process, then  $m \leq n$ . Otherwise the set  $\{w_{m-n}, w_{m-n+1}, \dots, w_{m-1}, w_m\}$  would be LD, which contradicts the linear independence of the  $w$ 's. ■

**3.6.3 Corollary** If  $V$  has a basis of  $n$  elements, then every set of  $p$  vectors with  $p > n$ , is LD.

**Proof:** Suppose  $B = \{v_1, v_2, \dots, v_n\}$  is a basis for  $V$ . Let  $A = \{u_1, u_2, \dots, u_p\}$  be a set of  $p$  vectors,  $p > n$ . If  $A$  is LI, then  $p \leq n$  by Theorem 3.6.2. Hence,  $A$  is LD. ■

**3.6.4 Corollary** If  $V$  has a basis of  $n$  elements, then every other basis for  $V$  also has  $n$  elements.

**Proof:** Suppose  $B_1 = \{v_1, v_2, \dots, v_n\}$  and  $B_2 = \{w_1, w_2, \dots, w_m\}$  are two bases for  $V$ . Then  $B_1$  and  $B_2$  are LI and  $[B_1] = V, [B_2] = V$ .

Since  $[B_1] = V$  and  $B_2$  is LI we have, by Theorem 3.6.2,  $m \leq n$ . Since  $[B_2] = V$  and  $B_1$  is LI we get, by the same theorem,  $n \leq m$ . Thus,  $m = n$ . ■

The number of elements in a basis is therefore unique and hence the following definitions.

**3.6.5 Definition** If a vector space  $V$  has a basis consisting of a finite number of elements, the space is said to be *finite-dimensional*; the number of elements in a basis is called the *dimension* of the space and is written as  $\dim V$ . If  $\dim V = n$ ,  $V$  is said to be *n-dimensional*. If  $V$  is not finite-dimensional, it is called *infinite-dimensional*.

If  $V = V_0 = \{0\}$ , its dimension is taken to be zero.

**3.6.6 Remark** It follows from Corollary 3.6.4 that, if a vector space  $V$  is  $n$ -dimensional, (a) there exist  $n$  linearly independent vectors in  $V$  and (b) every set of  $n + 1$  vectors in  $V$  is LD.

**Example 3.36**  $V_2$  is 2-dimensional, because  $(1, 0)$  and  $(0, 1)$  form a basis for  $V_2$ .  $V_3$  is 3-dimensional, because  $V_3$  has a basis of 3 elements, namely,  $i, j$ , and  $k$ .

$V_n$  is  $n$ -dimensional, for, consider the elements

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \dots, 1) \end{aligned}$$

with  $e_i$  as the vector, all of whose coordinates are zero except the  $i$ -th, which is 1. It is easy to see that  $e_1, e_2, \dots, e_n$  are LI and every  $n$ -tuple is a linear combination of  $e_1, e_2, \dots, e_n$ .

Thus, the set  $\{e_1, e_2, \dots, e_n\}$  is a basis for  $V_n$ . The basis  $\{e_1, e_2, \dots, e_n\}$  is called the *standard basis* for  $V_n$  and will be used in the sequel without any further mention. In particular, note that  $\{i, j, k\}$  is the standard basis for  $V_3$ . In other words,

$$\begin{aligned} e_1 &= (1, 0, 0) = i \\ e_2 &= (0, 1, 0) = j \\ e_3 &= (0, 0, 1) = k \end{aligned}$$

**Example 3.37** Find the dimension of the space  $\mathcal{P}_n$ .

Every polynomial in  $\mathcal{P}_n$  is a linear combination of the functions  $1, x, x^2, \dots, x^n$ . Further, this set is LI. (Why?) Hence  $\dim \mathcal{P}_n = n + 1$ .

**3.6.7 Theorem** In an  $n$ -dimensional vector space  $V$ , any set of  $n$  linearly independent vectors is a basis.

**Proof:** Suppose  $B = \{v_1, v_2, \dots, v_n\}$  is a set of  $n$  linearly independent vectors. To prove that  $B$  is a basis we have only to show that  $B$  spans  $V$ .

Take  $V \in V$ . The set  $\{v_1, v_2, \dots, v_n, v\}$  is LD, because  $V$  is  $n$ -dimensional (cf Remark 3.6.6). Hence, by Theorem 3.5.14, one of the vectors  $v_2, v_3, \dots, v_n$ , say  $v$ , is in the span of all its predecessors. Obviously, this one cannot be any one of  $v_2, v_3, \dots, v_n$ , for in that case we shall be contradicting the linear independence of  $\{v_1, v_2, \dots, v_n\}$ . Hence,  $v \in \{v_1, v_2, \dots, v_n\}$ . Therefore,  $B$  spans  $V$ . ■

**Example 3.38** Prove that the set  $\{(1, 1, 1), (1, -1, 1), (0, 1, 1)\}$  is a basis for  $V_3$ .

To prove this we have to prove only that the set is LI, because  $V_3$  is 3-dimensional. Suppose

$$\alpha(1, 1, 1) + \beta(1, -1, 1) + \gamma(0, 1, 1) = (0, 0, 0)$$

or 
$$(\alpha + \beta, \alpha - \beta + \gamma, \alpha + \beta + \gamma) = (0, 0, 0).$$

So  $\alpha + \beta = 0$ ,  $\alpha - \beta + \gamma = 0$ ,  $\alpha + \beta + \gamma = 0$ . These equations give  $\alpha = 0 = \beta = \gamma$ . Hence, the set is LI and is therefore a basis for  $V_3$ .

We shall now take up the theorem referred to immediately after Definition 3.6.1. This theorem gives rise to the definition of a basis used by some authors (see Remark 3.6.9).

**3.6.8 Theorem** *In a vector space  $V$  let  $B = \{v_1, v_2, \dots, v_n\}$  span  $V$ . Then the following two conditions are equivalent :*

(a)  $\{v_1, v_2, \dots, v_n\}$  is a linearly independent set.

(b) If  $v \in V$ , then the expression  $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$  is unique.

*Proof:* Assume (a). We shall prove that any expression for  $v$  in terms of  $v_1, v_2, \dots, v_n$  is unique. For, if

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \quad (1)$$

and also

$$v = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n, \quad (2)$$

then  $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$

or  $(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \dots + (\alpha_n - \beta_n)v_n = 0$ .

But the  $v_i$ 's are LI, so  $\alpha_i - \beta_i = 0$  for all  $i$ . This gives  $\alpha_i = \beta_i$  for all  $i$  and hence expression (1) is unique.

Conversely, assume (b). Suppose now that the  $v_i$ 's are not LI, i.e. they are LD. Then there exists a nontrivial linear combination, say

$$\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n, \quad (3)$$

which equals the zero vector. But 0, on the other hand, is already equal to the trivial linear combination

$$0v_1 + 0v_2 + \dots + 0v_n. \quad (4)$$

Thus, we get two different expressions, (3) and (4), for 0. This contradicts (b). Hence,  $\{v_1, v_2, \dots, v_n\}$  is LI. ■

**3.6.9 Remark** Note that the foregoing proof can be adopted for infinite-dimensional cases also.

Further, note that a basis  $B$  has two properties :

(i)  $B$  is LI and

(ii)  $[B] = V$ .

Hence, the following conclusion is obtained : 'A set  $B$  is a basis for a vector space  $V$  iff  $[B] = V$  and the expression for any  $v \in V$  in terms of elements of  $B$  is unique.'

We now define the term coordinate vector.

**3.6.10 Definition** Let  $B = \{v_1, v_2, \dots, v_n\}$  be an ordered basis for  $V$ . Then a vector  $v \in V$  can be written as  $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ . The vector  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  is called the *coordinate vector of  $v$  relative to the ordered basis  $B$* . It is denoted by  $[v]_B$ .  $\alpha_1, \alpha_2, \dots, \alpha_n$  are called the *coordinates* of vector  $v$  relative to the ordered basis  $B$ .

The coordinates of a vector relative to the standard basis are simply called the coordinates of the vector.

It should be noted that  $[v]_B$  is unambiguously fixed in view of Remark 3.6.9.

**Example 3.39** Find the coordinate vector of the vector  $(2, 3, 4, -1)$  of  $V_4$  relative to the standard basis for  $V_4$ .

The standard basis for  $V_4$  is  $\{e_1, e_2, e_3, e_4\}$ , where  $e_1 = (1, 0, 0, 0)$ ,  $e_2 = (0, 1, 0, 0)$ ,  $e_3 = (0, 0, 1, 0)$ , and  $e_4 = (0, 0, 0, 1)$ . Obviously,  $(2, 3, 4, -1) = 2e_1 + 3e_2 + 4e_3 - 1e_4$ . So the coordinate vector of  $(2, 3, 4, -1)$  relative to the standard basis is  $(2, 3, 4, -1)$ . Therefore, 2, 3, 4, and  $-1$  are the coordinates of the vector  $(2, 3, 4, -1)$ .

If the reader finds that this example is not very illuminating, he should study the following example.

**Example 3.40** Find the coordinates of  $(2, 3, 4, -1)$  relative to the ordered basis  $B = \{(1, 1, 0, 0), (0, 1, 1, 0), (0, 0, 1, 1), (1, 0, 0, 0)\}$  for  $V_4$ . (The reader can verify for himself that  $B$  is a basis for  $V_4$ .) To get the coordinates of  $(2, 3, 4, -1)$  relative to  $B$ , we write

$$(2, 3, 4, -1) = \alpha(1, 1, 0, 0) + \beta(0, 1, 1, 0) + \gamma(0, 0, 1, 1) + \delta(1, 0, 0, 0)$$

or 
$$(2, 3, 4, -1) = (\alpha + \delta, \alpha + \beta, \beta + \gamma, \gamma).$$

This gives  $\alpha + \delta = 2$ ,  $\alpha + \beta = 3$ ,  $\beta + \gamma = 4$ ,  $\gamma = -1$ . Solving these equations, we get  $\alpha = -2$ ,  $\beta = 5$ ,  $\gamma = -1$ , and  $\delta = 4$ . Hence, the coordinates of  $(2, 3, 4, -1)$  relative to the ordered basis  $B$  are  $-2, 5, -1$ , and 4.  $(-2, 5, -1, 4) = [(2, 3, 4, -1)]_B$ .

We now prove a theorem on the extension of a linearly independent set to a basis for a vector space  $V$ .

**3.6.11 Theorem** Let the set  $\{v_1, v_2, \dots, v_k\}$  be a linearly independent subset of an  $n$ -dimensional vector space  $V$ . Then we can find vectors  $v_{k+1}, \dots, v_n$  in  $V$  such that the set  $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$  is a basis for  $V$ .

**Proof:** By Theorem 3.6.2,  $k < n$ . If  $k = n$ , then, by Theorem 3.6.7,  $\{v_1, v_2, \dots, v_{k(-n)}\}$  is a basis for  $V$ . If  $k < n$ , then  $\{v_1, v_2, \dots, v_k\}$  is not a basis for  $V$  (Corollary 3.6.4). But the set  $\{v_1, v_2, \dots, v_k\}$  is LI. Therefore,  $[v_1, v_2, \dots, v_k] \neq V$ . Hence,  $[v_1, v_2, \dots, v_k]$  is a proper subset of  $V$ . Thus, there exists any nonzero vector  $v_{k+1}$  in  $V$  such that  $v_{k+1} \notin [v_1, v_2, \dots, v_k]$ .

Hence, the set  $\{v_1, v_2, \dots, v_k, v_{k+1}\}$  is LI; otherwise, by Theorem 3.5.14, one of the vectors  $v_2, v_3, \dots, v_k$ , say  $v_{k+1}$ , is in the span of all its predecessors.

Now, if  $k + 1 = n$ , we are done. If not, we repeat the foregoing process until we get  $n$  linearly independent vectors  $\{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ . This forms a basis for  $V$  by Theorem 3.6.7. ■

**3.6.12 Remark** We can produce any number of bases, because we can start from any nonzero vector and extend it.

*Example 3.41* Given two linearly independent vectors  $(1, 0, 1, 0)$  and  $(0, -1, 1, 0)$  of  $V_4$ , find a basis for  $V_4$  that includes these two vectors.  $[(1, 0, 1, 0), (0, -1, 1, 0)] = \{(\alpha, -\beta, \alpha + \beta, 0) \mid \alpha, \beta \text{ any scalars}\}$ .

As in Theorem 3.6.11, so too now we choose a vector outside this span and get an enlarged linearly independent set.

Since the fourth coordinate is always zero for vectors in this span, certainly  $(0, 0, 0, 1)$  is not in this span. Thus, we get an enlarged linearly independent set  $\{(1, 0, 1, 0), (0, -1, 1, 0), (0, 0, 0, 1)\}$ , whose span is  $[(1, 0, 1, 0), (0, -1, 1, 0), (0, 0, 0, 1)] = \{(\alpha, -\beta, \alpha + \beta, \gamma) \mid \alpha, \beta, \gamma \text{ are scalars}\}$ . Now we have to identify one element outside this span. Given  $\alpha, \beta, \gamma$ , the third coordinate in the elements of this span is always  $\alpha + \beta$ . So we shall look for a vector for which this is not true. Clearly,  $(1, -2, 0, 0)$  is not in the span of the earlier set. So we have a set

$$B = \{(1, 0, 1, 0), (0, -1, 1, 0), (0, 0, 0, 1), (1, -2, 0, 0)\},$$

which is LI and, by Theorem 3.6.7, it is a basis for  $V_4$ . The reader is advised to verify that this is indeed a basis.

*Example 3.42* Let  $\{(1, 1, 1, 1), (1, 2, 1, 2)\}$  be a linearly independent subset of the vector space  $V_4$ . Extend it to a basis for  $V_4$ .

We have

$$[(1, 1, 1, 1), (1, 2, 1, 2)] = \{(\alpha + \beta, \alpha + 2\beta, \alpha + \beta, \alpha + 2\beta) \mid \alpha, \beta \text{ are scalars}\}.$$

Since the first and third coordinates are equal for all vectors in the span, we find that  $(0, 3, 2, 3)$  is not in the span. Thus, we have an enlarged linearly independent set  $\{(1, 1, 1, 1), (1, 2, 1, 2), (0, 3, 2, 3)\}$ , whose span is

$$[(1, 1, 1, 1), (1, 2, 1, 2), (0, 3, 2, 3)] = \{(\alpha + \beta, \alpha + 2\beta + 3\gamma, \alpha + \beta + 2\gamma, \alpha + 2\beta + 3\gamma) \mid \alpha, \beta, \gamma \text{ are scalars}\}.$$

Obviously, the vector  $(2, 6, 4, 5)$  is not in this span. Hence, the set  $\{(1, 1, 1, 1), (1, 2, 1, 2), (0, 3, 2, 3), (2, 6, 4, 5)\}$  is LI. Thus, by Theorem 3.6.7, it is a basis for  $V_4$ .

We shall now prove two theorems on dimensions of subspaces.

**3.6.13 Theorem** Let  $U$  be a subspace of a finite-dimensional vector space  $V$ . Then  $\dim U \leq \dim V$ . Equality holds only when  $U = V$ .

*Proof:* Let  $B = \{v_1, v_2, \dots, v_n\}$  be a basis for  $V$ . This generates  $V$  and has  $n$  elements. Any set of linearly independent vectors in  $V$  and therefore any set of linearly independent vectors in  $U$  cannot have more than  $n$  vectors. Therefore,  $\dim U \leq \dim V$ .

When  $\dim U = \dim V$ ,  $B_1$ , a basis for  $U$ , is a set of  $n$  linearly independent vectors of  $V$ , whose dimension is also  $n$ . So, by Theorem 3.6.5, it follows that  $B_1$  is a basis for  $V$ . This means  $V = [B_1] = U$ .  $\blacksquare$

**3.6.14 Theorem** *If  $U$  and  $W$  are two subspaces of a finite-dimensional vector space  $V$ , then*

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

*Proof:* Let  $\dim U = m$ ,  $\dim W = p$ ,  $\dim(U \cap W) = r$ , and  $\dim V = n$ . By Theorem 3.6.13,  $m \leq n$ ,  $p \leq n$ ,  $r \leq n$ . Let  $\{v_1, v_2, \dots, v_r\}$  be a basis for  $U \cap W$ . This is a linearly independent set in  $U \cap W$  and therefore in  $U$  as well as in  $W$ . So it can be extended to a basis for  $U$ , say

$$\{v_1, v_2, \dots, v_r, u_{r+1}, \dots, u_m\} \quad (5)$$

and to a basis for  $W$ , say

$$\{v_1, v_2, \dots, v_r, w_{r+1}, \dots, w_p\}. \quad (6)$$

We shall now prove that the set

$$A = \{v_1, v_2, \dots, v_r, u_{r+1}, \dots, u_m, w_{r+1}, \dots, w_p\} \quad (7)$$

is a basis for  $U + W$ . In fact, this will complete the proof of the theorem, because in that case, the dimension of  $U + W$  will be the number of elements in  $A$ , i.e.  $r + (m - r) + (p - r) = m + p - r$ . This is what the theorem asserts.

To prove that  $A$  is a basis for  $U + W$ , we have to prove that (a)  $A$  is LI in  $U + W$  and (b)  $[A] = U + W$ . To prove (a) let us assume that

$$\sum_{i=1}^r \alpha_i v_i + \sum_{i=r+1}^m \beta_i u_i + \sum_{i=r+1}^p \gamma_i w_i = 0. \quad (8)$$

This gives

$$\sum_{i=1}^r \alpha_i v_i + \sum_{i=r+1}^m \beta_i u_i = - \sum_{i=r+1}^p \gamma_i w_i \quad (9)$$

$= v$ , say.

The vector  $v$  is in  $U$ , because the left-hand side of Equation (9) is in  $U$ .  $v$  is also in  $W$ , since the right-hand side of Equation (9) is in  $W$ . Thus,  $v \in U \cap W$ . Therefore,  $v$  can be expressed uniquely in terms of  $v_1, v_2, \dots, v_r$ . Thus,

$$v = \sum_{i=1}^r \delta_i v_i \quad (10)$$

for suitable  $\delta$ 's. Hence,

$$\sum_{i=1}^r \delta_i v_i + \sum_{i=r+1}^p \gamma_i w_i = 0. \quad (11)$$



But  $\{v_1, v_2, \dots, v_r, w_{r+1}, \dots, w_p\}$  is LI. So each of the  $\delta_i$ 's and  $\gamma_i$ 's is zero. Substituting  $\gamma_{r+1} = \gamma_{r+2} = \dots = \gamma_p = 0$  in Equation (9), we find that

$$\sum_{i=1}^r \alpha_i v_i + \sum_{i=r+1}^m \beta_i u_i = 0. \quad (12)$$

Again,  $\{v_1, \dots, v_r, u_{r+1}, \dots, u_m\}$  is LI. So each of the  $\alpha_i$ 's and  $\beta_i$ 's is zero. Thus, Equation (8) implies that each scalar involved is zero. Hence,  $A$  is LI, which proves (a).

To prove (b), let  $z \in U + W$ . Then  $z = u + w$ , where  $u \in U$  and  $w \in W$ . This gives

$$z = \sum_{i=1}^r \alpha_i v_i + \sum_{i=r+1}^m \beta_i u_i + \sum_{i=1}^r \alpha'_i v_i + \sum_{i=r+1}^p \beta'_i w_i, \quad (13)$$

for suitable scalars  $\alpha_i$ 's,  $\beta_i$ 's,  $\alpha'_i$ 's,  $\beta'_i$ 's. Simplifying expression (13), we see that  $z \in [A]$ . Hence,  $U + W \subset [A]$ . The reverse in equality is obvious, because  $A \subset U + W$ . ■

**3.6.15 Corollary** *If  $U$  and  $W$  are subspaces of a finite-dimensional vector space  $V$  such that  $U \cap W = \{0\}$ , then*

$$\dim(U \oplus W) = \dim U + \dim W.$$

The proof is obvious and is left to the reader.

As illustration, take  $U =$  the  $xy$ -plane and  $W =$  the  $yz$ -plane in  $V_3$ . Clearly,  $U$  and  $W$  are subspaces of  $V_3$  and  $\dim U = 2$ ,  $\dim W = 2$ .  $U \cap W = y$ -axis, whose dimension is 1. By Theorem 3.6.14, we have

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

But  $U + W$ , in this case, is  $V_3$ . So

$$\dim V_3 = (\text{xy-plane}) + \dim(\text{yz-plane}) - \dim(\text{y-axis})$$

$$\text{or} \quad 3 = 2 + 2 - 1.$$

This verifies the result of Theorem 3.6.14.

On the other hand, if we take  $U = xy$ -plane and  $W = z$ -axis, then  $U \cap W = \{0\}$  and  $U + W = V_3$ . Also,

$$\dim V_3 = \dim(U \oplus W) = \dim U + \dim W,$$

$$\text{or} \quad 3 = 2 + 1.$$

This verifies the result of Corollary 3.6.15.

Now we give a comprehensive example which illustrates different ideas studied in this chapter.

**Example 3.43** Take  $V = \mathcal{P}_3$ , the space of all real polynomials of degree at most 3. Let  $U$  be the subspace of  $\mathcal{P}_3$  consisting of those polynomials of  $\mathcal{P}_3$  that vanish at  $x = 1$ . Let  $W$  be the subspace of  $\mathcal{P}_3$  consisting of those polynomials of  $\mathcal{P}_3$  whose first derivatives vanish at  $x = 1$ . Study the subspaces  $U$ ,  $W$ ,  $U \cap W$ ,  $U + W$  in terms of dimension, basis, and the extension of each of these bases to a basis for  $V = \mathcal{P}_3$ .

Let  $p$  stand for an arbitrary polynomial. Recall that  $R$  stands for the set of real numbers. Then

$$\begin{aligned} V &= \{p \mid \text{degree of } p < 3\} \\ &= \{\alpha + \beta x + \gamma x^2 + \delta x^3 \mid \alpha, \beta, \gamma, \delta \in R\}. \end{aligned}$$

We know that  $\dim V = 4$ . Also

$$\begin{aligned} U &= \{p \in V \mid p(1) = 0\} \\ &= \{\alpha + \beta x + \gamma x^2 + \delta x^3 \mid \alpha + \beta + \gamma + \delta = 0; \\ &\quad \alpha, \beta, \gamma, \delta \in R\} \\ &= \{(-\beta - \gamma - \delta) + \beta x + \gamma x^2 + \delta x^3 \mid \beta, \gamma, \delta \in R\} \\ &= \{\beta(x - 1) + \gamma(x^2 - 1) + \delta(x^3 - 1) \mid \beta, \gamma, \delta \in R\}. \end{aligned}$$

This shows that the set  $\{x - 1, x^2 - 1, x^3 - 1\}$  spans  $U$ . Since it is easily seen that this set is LI, it also forms a basis for  $U$ . So  $\dim U = 3$ .

This basis can be extended to a basis for  $V = \mathcal{P}_3$  by taking another element that is not in the span  $U$  of these polynomials. Such a polynomial is a constant polynomial, say 1. Thus, by Theorem 3.6.7,  $\{1, x - 1, x^2 - 1, x^3 - 1\}$  is a basis for  $V = \mathcal{P}_3$ . Again,

$$\begin{aligned} W &= \{p \in V \mid p'(1) = 0\} \\ &= \{\alpha + \beta x + \gamma x^2 + \delta x^3 \mid \beta + 2\gamma + 3\delta = 0; \alpha, \beta, \gamma, \delta \in R\} \\ &= \{\alpha + \gamma(x^2 - 2x) + \delta(x^3 - 3x) \mid \alpha, \gamma, \delta \in R\}. \end{aligned}$$

This shows that  $W$  is spanned by  $\{1, x^2 - 2x, x^3 - 3x\}$ . Since this can be shown to be LI, it follows that  $\{1, x^2 - 2x, x^3 - 3x\}$  is a basis for  $W$  and  $\dim W = 3$ .

The extension of this basis to a basis for  $\mathcal{P}_3$  is got by looking for just one vector that is not in  $W$ . Such a vector is a polynomial  $x$ . Thus,  $\{1, x, x^2 - 2x, x^3 - 3x\}$  is a basis for  $\mathcal{P}_3$ . Again,

$$\begin{aligned} U \cap W &= \{p \in \mathcal{P}_3 \mid p(1) = 0, p'(1) = 0\} \\ &= \{\alpha + \beta x + \gamma x^2 + \delta x^3 \mid \alpha + \beta + \gamma + \delta = 0, \\ &\quad \beta + 2\gamma + 3\delta = 0; \alpha, \beta, \gamma, \delta \in R\} \\ &= \{\gamma(1 - 2x + x^2) + \delta(2 - 3x + x^3) \mid \gamma, \delta \in R\}. \end{aligned}$$

Clearly,  $1 - 2x + x^2$  and  $2 - 3x + x^3$  are LI. So  $\dim U \cap W = 2$  and  $\{1 - 2x + x^2, 2 - 3x + x^3\}$  is a basis for  $U \cap W$ . Now we look for elements that are not in  $U \cap W$ . One such element is  $x^3$ . Now consider the span of the linearly independent set  $\{1 - 2x + x^2, 2 - 3x + x^3, x^3\}$ . The span of this set is still not  $V$ , because  $\dim V = 4$ . So look for one more element outside this span. One such element is  $x^2$ . (Why? Can you find others?) Hence,

$$\{1 - 2x + x^2, 2 - 3x + x^3, x^2, x^3\}$$

is a basis for  $V$ . Finally,

$$\begin{aligned} U + W &= \{p + q \mid p \in U, q \in W\} \\ &= \{\alpha_1(x - 1) + \alpha_2(x^2 - 1) + \alpha_3(x^3 - 1) + \beta_1 \\ &\quad + \beta_2(x^2 - 2x) + \beta_3(x^3 - 3x) \mid \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in R\}. \end{aligned}$$

So  $B = \{(x-1), (x^2-1), (x^3-1), 1, (x^2-2x), (x^3-3x)\}$  spans the space  $U + W$ . But

$$\begin{aligned}\dim(U + W) &= \dim U + \dim W - \dim(U \cap W) \\ &= 3 + 3 - 2 = 4.\end{aligned}$$

Therefore,  $B$  is LD. Applying Theorem 3.5.9 to the ordered set  $C = \{1, x-1, x^2-1, x^2-2x, x^3-1, x^3-3x\}$ , which is LD, we find that  $x^3-2x \in [1, x-1, x^2-1]$ , for

$$x^3-2x = 1(x^2-1) - 2(x-1) - 1(1).$$

Hence,  $x^3-2x$  can be discarded. We are thus left with the set

$$C' = \{1, x-1, x^2-1, x^3-1, x^3-3x\},$$

which has 5 elements. This is still LD (since  $\dim(U + W) = 4$ ). Again, by Theorem 3.5.9, there exists a vector that is a linear combination of all its predecessors. It cannot be 1 or  $x-1$  or  $x^2-1$  or  $x^3-1$ , since these four are LI among themselves. So let us check whether  $x^3-3x$  is a linear combination of its predecessors. We have

$$x^3-3x = 1(x^3-1) + 0(x^2-1) - 3(x-1) - 2(1).$$

Hence,  $x^3-3x \in [1, x-1, x^2-1, x^3-1]$ . So  $x^3-3x$  can be discarded. We are thus left with the set

$$D = \{1, x-1, x^2-1, x^3-1\},$$

which is obviously a basis for  $U + W$ . Since  $\dim(U + W) = \dim V = 4$ , we get  $U + W = V$ , by Theorem 3.6.13.

### Problem Set 3.6

- Which of the following subsets  $S$  form a basis for  $V_3$ ? In case  $S$  is not a basis for  $V_3$ , determine a basis for  $[S]$ .
  - $S = \{(1, 2, 3), (3, 1, 0), (-2, 1, 3)\}$
  - $S = \{(1, 1, 1), (1, 2, 3), (-1, 0, 1)\}$
  - $S = \{(0, 0, 1), (1, 0, 1), (1, -1, 1), (3, 0, 1)\}$
  - $S = \{(1, 2/5, -1), (0, 1, 2), (3/4, -1, 1)\}$
  - $S = \{(-1, 3/2, 2), (3, 2/3, 3)\}$
- Which of the following subsets  $S$  form a basis for the given vector space  $V$ ?
  - $S = \{(1, -1, 0, 1), (0, 0, 0, 1), (2, -1, 0, 1), (3, 2, 1, 0)\}$ ,  
 $V = V_4$
  - $S = \{(0, 1, 2, 1), (1, 2, -1, 1), (2, -3, 1, 0), (4, -2, -7, -5)\}$ ,  
 $V = V_4$
  - $S = \{x-1, x^2+x-1, x^2-x+1\}$ ,  $V = \mathcal{P}_3$
  - $S = \{1, x, (x-1)x, x(x-1)(x-2)\}$ ,  $V = \mathcal{P}_3$
  - $S = \{1, x, (3x^2-1)/2, (5x^2-3x)/2, (35x^4-30x^3+3)/8\}$ ,  
 $V = \mathcal{P}_4$
  - $S = \{1, x-2, (x-2)^2, (x-2)^3\}$ ,  $V = \mathcal{P}_3$

- (g)  $S = \{1, \sin x, \sin^2 x, \cos^2 x\}$ ,  $V = \mathcal{C}[-\pi, \pi]$   
 (h)  $S = \{(1, i, 1 + i), (1, i, 1 - i), (i, -i, 1)\}$ ,  $V = V_3^{\mathbb{C}}$ .
- Determine the dimension of the subspace  $[S]$  of  $V_3$  for each  $S$  in Problem 1.
  - Determine the dimension of the subspace  $[S]$  of  $V$  for each  $S$  in Problem 2.
  - Extend the set  $\{(3, -1, 2)\}$  to two different bases for  $V_3$ .
  - Given  $S_1 = \{(1, 2, 3), (0, 1, 2), (3, 2, 1)\}$  and  $S_2 = \{(1, -2, 3), (-1, 1, -2), (1, -3, 4)\}$ , determine the dimension and a basis for
    - $[S_1] \cap [S_2]$
    - $[S_1] + [S_2]$ .
  - Given  $S$  as a finite subset of a vector space  $V$ , prove that
    - If  $S$  is LI and every proper superset of  $S$  in  $V$  is LD, then  $S$  is a basis for  $V$ .
    - If  $S$  spans  $V$  and no proper subset of  $S$  spans  $V$ , then  $S$  is a basis for  $V$ .
  - Find a basis for a subspace  $U$  of  $V$  in the following cases :
    - $U = \{p \in \mathcal{P}_5 \mid p'' = 0\}$ ,  $V = \mathcal{P}_5$
    - $U = \{p \in \mathcal{P}_5 \mid p(x_0) = p'(x_0) = 0\}$ ,  $V = \mathcal{P}_5$
    - $U = \{(x_1, x_2, \dots, x_n) \in V_n \mid \alpha_1 x_1 + \dots + \alpha_n x_n = 0; \alpha_1, \dots, \alpha_n \text{ are any } n \text{ scalars}\}$ ,  $V = V_n$
    - $U = \{(x_1, x_2, x_3, x_4, x_5) \in V_5 \mid \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{matrix}\}$ ,  
 $V = V_5$
    - $U = \{p \in \mathcal{P}_4 \mid p(x_0) = 0\}$ ,  $V = \mathcal{P}_4$ .
  - Let  $U$  and  $W$  be two distinct  $(n - 1)$ -dimensional subspaces of an  $n$ -dimensional vector space  $V$ . Then prove that  $\dim(U \cap W) = n - 2$ .
  - Prove that every 1-dimensional subspace of  $V_3$  is a straight line through the origin.
    - Prove that every 2-dimensional subspace of  $V_3$  is a plane through the origin.
    - Deduce that the intersection of two distinct planes through the origin is a straight line through the origin.
  - Let  $v_i = \alpha_i e_k - \alpha_k e_i$ ,  $i = 1, 2, \dots, n$ , and for a fixed  $k (< n)$ , be  $n$  vectors of  $V_n$  with  $\alpha_k \neq 0$ . Then prove that  $v_1, v_2, \dots, v_{k-1}, v_{k+1}, \dots, v_n$  is a basis for  $U$  of Problem 8(c).
  - Find the coordinates of the following vectors of  $V_3$  relative to the ordered basis  $B = \{(2, 1, 0), (2, 1, 1), (2, 2, 1)\}$  :
    - $(1, 2, 1)$
    - $(-1, 3, 1)$
    - $(x_1, x_2, x_3)$
    - $(-\sqrt{2}, \pi, e)$
    - $(-1/2, 11/3, 5)$
    - $(2, 0, -1)$ .

13. Find the coordinates of the following polynomials relative to the ordered basis  $\{1 - x, 1 + x, 1 - x^2\}$  of  $\mathcal{P}_2$ :
- (a)  $3 + 7x + 2x^2$     (b)  $x - 3x^2$     (c)  $x^2 + 2x - 1$ .
14. Find an ordered basis for  $V_4$  relative to which the vector  $(-1, 3, 2, 1)$  has the coordinates 4, 1,  $-2$ , and 7.
15. Let  $B_1 = \{v_1, v_2, \dots, v_n\}$  and  $B_2 = \{u_1, u_2, \dots, u_n\}$  be two ordered bases for an  $n$ -dimensional vector space  $V$  and let

$$u_1 = \alpha_{11}v_1 + \alpha_{21}v_2 + \dots + \alpha_{n1}v_n$$

$$u_2 = \alpha_{12}v_1 + \alpha_{22}v_2 + \dots + \alpha_{n2}v_n$$

$$\vdots$$

$$u_n = \alpha_{1n}v_1 + \alpha_{2n}v_2 + \dots + \alpha_{nn}v_n.$$

Then find the coordinates of a vector  $v \in V$  relative to the basis  $B_1$ , if its coordinates relative to the basis  $B_2$  are  $x_1, x_2, \dots, x_n$ .

16. Find the dimension of  $V_2^\cap$ .
17. Construct two subspaces  $A$  and  $B$  of  $V_4$  such that  $\dim A = 2$ ,  $\dim B = 3$ , and  $\dim A \cap B = 1$ .
18. Let  $B_1 = \{u_1, u_2, \dots, u_n\}$  and  $B_2 = \{v_1, v_2, \dots, v_n\}$  be ordered bases for an  $n$ -dimensional vector space  $V$  such that  $\{u_1 - v_1, u_2 - v_2, \dots, u_n - v_n\}$  is LD. Then prove that there exists a nonzero vector  $u \in V$  such that  $[u]_{B_1} = [u]_{B_2}$ .
19. True or false?
- (a) Every vector space has a finite basis.
  - (b) A basis can never include the zero vector.
  - (c) If two bases of  $V$  have one common vector, then the two bases are the same.
  - (d) If every set of  $p$  vectors of  $V$  with  $p > n$  is LD, then  $V$  has a set of  $n$  generators.
  - (e) A basis for  $V_3$  can be extended to a basis for  $V_4$ .
  - (f) A basis for  $V_3$  is  $\{i + j + k, i + j, i\}$ .
  - (g) A basis for  $\mathcal{P}_3$  is  $\{1, 2x, (x - 1)^2\}$ .
  - (h) A set of generators of  $\mathcal{P}_3$  is  $\{(x - 1)^3, (x - 1)^2, (x - 1), x^2 - 1, 1\}$ .
  - (i)  $[v]_B$  is independent of  $B$ .
  - (j)  $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  is a basis for the complex vector space  $V_3^C$ .
  - (k) In  $V_3$ , if  $(2, 3)$  is the coordinate vector of  $(2, 3)$  relative to the ordered basis  $B$ , then  $B$  is the standard basis.

# Linear Transformations

The significance of vector spaces arises from the fact that we can pass from one vector space to another by means of functions that possess a certain property called linearity. These functions are called linear transformations.

## 4.1 DEFINITION AND EXAMPLES

**4.1.1 Definition** Suppose  $U$  and  $V$  are vector spaces either both real or both complex. Then the map  $T: U \rightarrow V$  is said to be a *linear map* (transformation, operator), if

$$T(u_1 + u_2) = T(u_1) + T(u_2) \text{ for all } u_1, u_2 \in U \quad (1)$$

$$\text{and} \quad T(\alpha u) = \alpha T(u) \text{ for all } u \in U, \text{ and all scalars } \alpha. \quad (2)$$

A linear map  $T: U \rightarrow U$  is also called a linear map on  $U$ . Whenever we say  $T: U \rightarrow V$  is a linear map, then  $U$  and  $V$  shall be taken as vector spaces over the same field of scalars.

**4.1.2 Remark** In Definition 4.1.1 the 'plus' in  $u_1 + u_2$  denotes the addition in space  $U$ , and the 'plus' in  $T(u_1) + T(u_2)$  denotes the addition in space  $V$ . We shall not elaborate these delicate points in the sequel. A similar remark is true for the two scalar multiplications implied in Equation (2).

**Example 4.1** Define  $T: V_3 \rightarrow V_3$  by the rule

$$T(x_1, x_2, x_3) = (x_1, x_2, 0).$$

It is called the projection of  $V_3$  on the  $x_1x_2$ -plane. To prove that it is a linear map, we have to show that

$$T(x + y) = T(x) + T(y) \text{ and } T(\alpha x) = \alpha T(x)$$

for all  $x, y \in V_3$  and all scalars  $\alpha$ . Let  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$ . Then

$$x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

and

$$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3).$$

Now, by definition of  $T$ , we have

$$\begin{aligned} T(x + y) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1, x_2 + y_2, 0). \end{aligned}$$

On the other hand,

$$\begin{aligned} T(x) + T(y) &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) \\ &= (x_1, x_2, 0) + (y_1, y_2, 0) \quad (\text{definition of } T) \\ &= (x_1 + y_1, x_2 + y_2, 0) \quad (\text{addition in } V_3). \end{aligned}$$

Thus,  $T(x + y) = T(x) + T(y)$  for all  $x, y \in V_3$ .

This verifies one requirement for the linearity of  $T$ . Again,

$$\begin{aligned} T(\alpha x) &= T(\alpha x_1, \alpha x_2, \alpha x_3) \\ &= (\alpha x_1, \alpha x_2, 0) \quad (\text{definition of } T) \\ &= \alpha(x_1, x_2, 0) \quad (\text{scalar multiplication in } V_3) \end{aligned}$$

and

$$\alpha T(x) = \alpha T(x_1, x_2, x_3) = \alpha(x_1, x_2, 0) \quad (\text{definition of } T).$$

Thus,  $T(\alpha x) = \alpha T(x)$  for all  $x \in V_3$  and all scalars  $\alpha$ .

Hence, both the conditions for linearity of  $T$  are verified. So  $T$  is linear.

The reader should note that we have gone through the verification rather leisurely. We shall not be able to afford such a leisurely pace in the sequel. But every time we say that a map is linear the student would do well not to take it for granted but rather to go through the verification in as much detail as his understanding demands.

**Example 4.2** Define  $T: V_3 \rightarrow V_2$  by the rule

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3).$$

This is a linear map as we shall now show.

If  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$ , then

$$\begin{aligned} T(x + y) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1 - x_2 - y_2, x_1 + y_1 + x_3 + y_3), \end{aligned}$$

whereas

$$\begin{aligned} T(x) + T(y) &= (x_1 - x_2, x_1 + x_3) + (y_1 - y_2, y_1 + y_3) \\ &= (x_1 - x_2 + y_1 - y_2, x_1 + x_3 + y_1 + y_3) \\ &= (x_1 + y_1 - x_2 - y_2, x_1 + y_1 + x_3 + y_3). \end{aligned}$$

Hence,  $T(x + y) = T(x) + T(y)$  for all  $x, y \in V_3$ . The remaining part, namely,  $T(\alpha x) = \alpha T(x)$  can be completed by the reader. Then it will follow that  $T$  is linear.

**Example 4.3** Define  $T: V_3 \rightarrow V_1$  by the rule

$$T(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2.$$

This is *not* a linear map, because for  $x = y = (1, 0, 0)$

$$T(x + y) = (x_1 + y_1)^2 + (x_2 + y_2)^2 + (x_3 + y_3)^2 = 4,$$

whereas

$$T(x) + T(y) = x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 = 2.$$

Note that even if one of the two requirements of the definition of linearity fails,  $T$  is not linear.

**Example 4.4** Define  $T: U \rightarrow V$  ( $U$  and  $V$  being vector spaces) by the rule

$$T(u) = 0_V$$

for all  $u \in U$ . This is a linear map. It is called the *zero map*, because it maps every vector to the zero vector of  $V$ .

**Example 4.5** Let  $U$  be a vector space. The *identity map*  $I_U: U \rightarrow U$  defined by the rule

$$I_U(u) = u$$

is also linear.

**Example 4.6** Define  $T: V_2 \rightarrow V_2$  by the rule

$$T(x_1, x_2) = (x_1, -x_2).$$

This is a linear map and is called the *reflection* in the  $x_1$ -axis (Figure 4.1).

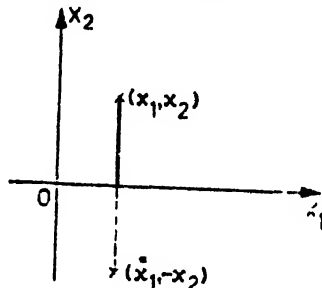


FIGURE 4.1

**Example 4.7** Define  $D: \mathcal{C}^1(a, b) \rightarrow \mathcal{C}(a, b)$  by the rule

$$D(f) = f',$$

where  $f'$  is the derivative of  $f$ . The facts that  $D(f + g) = (f + g)' = f' + g'$  and that  $D(\alpha f) = (\alpha f)' = \alpha f'$  are elementary but important results in calculus. In view of these results, we see that  $D$  is linear. This transformation  $D$  is called the *differential operator*.

**Example 4.8** Define  $\mathcal{J}: \mathcal{C}(a, b) \rightarrow \mathcal{R}$  by the rule

$$\mathcal{J}(f) = \int_a^b f(x) dx.$$

Again, by the properties of integral, we find that  $\mathcal{J}$  is a linear map.

**Example 4.9** Suppose  $u_0 \neq 0$  is a fixed vector of a vector space  $U$ . Define  $T: U \rightarrow U$  by the rule

$$T(x) = x + u_0$$

for all  $x \in U$ . This map  $T$  is *not* linear, because

$$\begin{aligned} T(x + y) &= (x + y) + u_0 \\ &\neq (x + u_0) + (y + u_0) = T(x) + T(y). \end{aligned}$$

This map is called *translation* by the vector  $u_0$ .

**4.1.3 Remark** The function  $f: \mathcal{R} \rightarrow \mathcal{R}$  defined by  $f(x) = x + a$  ( $a$  fixed) is customarily called a *linear function*, because its graph in the  $xy$ -



plane is a straight line. But it is not a linear map from the vector space  $V_1$  to itself, in the sense of Definition 4.1.1.

**4.1.4 Theorem** *Let  $T : U \rightarrow V$  be a linear map. Then*

- (a)  $T(0_U) = 0_V$ , (b)  $T(-u) = -T(u)$ , and  
 (c)  $T(\alpha_1 u_1 + \dots + \alpha_n u_n) = \alpha_1 T(u_1) + \dots + \alpha_n T(u_n)$ .

In other words, a linear map  $T$  transforms the zero of  $U$  into the zero of  $V$  and the negative of every  $u \in U$  into the negative of  $T(u)$  in  $V$ .

The proofs of (a) and (b) are left to the reader. The proof of (c) is obtained by induction on 'n', starting from the fact that  $T(\alpha u) = \alpha T(u)$  and using the property

$$T(\alpha u_1 + \beta u_2) = T(\alpha u_1) + T(\beta u_2) = \alpha T(u_1) + \beta T(u_2).$$

In view of (c), we get a standard technique of defining a linear transformation  $T$  on a finite-dimensional vector space. Suppose  $B = \{u_1, u_2, \dots, u_n\}$  is a basis for  $U$ . Then any vector  $u \in U$  can be expressed uniquely in the form

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

So, if  $T : U \rightarrow V$  is a linear map, then

$$\begin{aligned} T(u) &= T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) \\ &= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n). \end{aligned}$$

Thus,  $T(u)$  is known as soon as  $T(u_1), T(u_2), \dots, T(u_n)$  are known. This is formalised in the following theorem.

**4.1.5 Theorem** *A linear transformation  $T$  is completely determined by its values on the elements of a basis. Precisely, if  $B = \{u_1, u_2, \dots, u_n\}$  is a basis for  $U$  and  $v_1, v_2, \dots, v_n$  be  $n$  vectors (not necessarily distinct) in  $V$ , then there exists a unique linear transformation*

$$T : U \rightarrow V$$

*such that*

$$T(u_i) = v_i \text{ for } i = 1, 2, \dots, n. \quad (3)$$

*Proof:* Let  $u \in U$ . Then  $u$  can be expressed uniquely in the form

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

We define

$$T(u) = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n. \quad (4)$$

We now claim that this transformation  $T$  is the required transformation. To prove our claim, we have to show that (i)  $T$  is linear, (ii)  $T$  satisfies Equation (3), and (iii)  $T$  is unique.

(ii) is obvious, since  $u_i = 0u_1 + \dots + 0u_{i-1} + 1u_i + \dots + 0u_n$ , and so  $T(u_i) = 1v_i = v_i$  for all  $i$ .

(iii) follows, because if there were another such linear map  $S$  with  $S(u_i) = v_i$ , then

$$\begin{aligned} S(u) &= S(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) \\ &= \alpha_1 S(u_1) + \alpha_2 S(u_2) + \dots + \alpha_n S(u_n) \quad (S \text{ is linear}) \\ &= \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \\ &= T(u). \end{aligned}$$

This is true for every  $u \in U$ . So  $S = T$ .

It only remains to prove (1), which is just a verification of the two relations

$$T(u + v) = T(u) + T(v) \text{ and } T(\alpha u) = \alpha T(u)$$

for arbitrary  $u, v \in U$  and all scalars  $\alpha$ . Let  $u, v$  be two vectors of  $U$ . Then

$$u = \alpha_1 u_1 + \dots + \alpha_n u_n, v = \beta_1 u_1 + \dots + \beta_n u_n$$

and we have

$$u + v = (\alpha_1 + \beta_1)u_1 + \dots + (\alpha_n + \beta_n)u_n.$$

Hence, by the definition of  $T$ , we have

$$T(u + v) = (\alpha_1 + \beta_1)v_1 + \dots + (\alpha_n + \beta_n)v_n.$$

$$\begin{aligned} \text{Also, } T(u) + T(v) &= (\alpha_1 v_1 + \dots + \alpha_n v_n) + (\beta_1 v_1 + \dots + \beta_n v_n) \\ &= (\alpha_1 + \beta_1)v_1 + \dots + (\alpha_n + \beta_n)v_n. \end{aligned}$$

Therefore,  $T(u + v) = T(u) + T(v)$ . Again,

$$\begin{aligned} T(\alpha u) &= \alpha \alpha_1 v_1 + \dots + \alpha \alpha_n v_n \\ &= \alpha(\alpha_1 v_1 + \dots + \alpha_n v_n) \\ &= \alpha T(u). \end{aligned}$$

Thus,  $T$  is linear and the theorem is proved. ■

**4.1.6 Remark** Theorem 4.1.5 will be used in the following way: To define a linear map, very often we shall be content with listing  $T(u_1), T(u_2), \dots, T(u_r)$ . This means the value of  $T$  on a general  $u$  is to be obtained by the process shown in the theorem, namely, if

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n,$$

then

$$T(u) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n).$$

This process of defining  $T$  on the basis and extending it to the remaining elements is called *linear extension of  $T$* . So we shall simply say 'Define  $T$  on a basis and extend it linearly' or, equivalently, 'Define the linear map  $T$  by specifying its values on a basis'.

**Example 4.10** Suppose we want to define a linear map  $T: V_3 \rightarrow V_4$ . Take a basis for  $V_3$ , say  $\{(1, 1), (1, -1)\}$ . We have only to fix  $T(1, 1)$  and  $T(1, -1)$  in  $V_4$ . In fact, every ordered pair of vectors in  $V_4$  will give us one linear map  $T$ . We shall cite a few in Table 4.1.

TABLE 4.1

Linear map $\rightarrow$ Value at $\downarrow$	$T_1$	$T_2$	$T_3$	$T_4$
$(1, 1)$	$(0, 0, 0, 0)$	$(0, 1, 0, 0)$	$(1, 1, 1, 1)$	$(1, 1, 0, 0)$
$(1, -1)$	$(0, 0, 0, 0)$	$(1, 0, 0, 0)$	$(-1, -1, -1, -1)$	$(0, 0, 0, 0)$

Let us now linearly extend each of these maps to any  $u \in V_1$ . The actual values of  $T(u)$  will depend on the coordinates of  $u$  relative to the basis  $\{(1, 1), (1, -1)\}$ .

Let  $u = (x, y) \in V_1$ . Then

$$(x, y) = \frac{x+y}{2} (1, 1) + \frac{x-y}{2} (1, -1).$$

Thus, we get

$$\begin{aligned} T_1(x, y) &= \frac{x+y}{2} T_1(1, 1) + \frac{x-y}{2} T_1(1, -1) \\ &= \frac{x+y}{2} (0, 0, 0, 0) + \frac{x-y}{2} (0, 0, 0, 0) = (0, 0, 0, 0) \\ &= 0. \end{aligned}$$

Therefore,  $T_1$  is the zero map.

$$\begin{aligned} T_2(x, y) &= \frac{x+y}{2} T_2(1, 1) + \frac{x-y}{2} T_2(1, -1) \\ &= \frac{x+y}{2} (0, 1, 0, 0) + \frac{x-y}{2} (1, 0, 0, 0) \\ &= \left( \frac{x-y}{2}, \frac{x+y}{2}, 0, 0 \right). \end{aligned}$$

$$\begin{aligned} T_3(x, y) &= \frac{x+y}{2} T_3(1, 1) + \frac{x-y}{2} T_3(1, -1) \\ &= \frac{x+y}{2} (1, 1, 1, 1) + \frac{x-y}{2} (-1, -1, -1, -1) \\ &= (y, y, y, y). \end{aligned}$$

$$\begin{aligned} T_4(x, y) &= \frac{x+y}{2} T_4(1, 1) + \frac{x-y}{2} T_4(1, -1) \\ &= \frac{x+y}{2} (1, 1, 0, 0) + \frac{x-y}{2} (0, 0, 0, 0) \\ &= \left( \frac{x+y}{2}, \frac{x+y}{2}, 0, 0 \right). \end{aligned}$$

### Problem Set 4.1

- Let  $U$  and  $V$  be vector spaces over the same field of scalars, and  $T$  a map from  $U$  to  $V$ . Then prove that  $T$  is linear iff  $T(\alpha u_1 + u_2) = \alpha T(u_1) + T(u_2)$  for all  $u_1, u_2 \in U$  and scalar  $\alpha$ .
- Which of the following maps are linear?
  - $T: V_1 \rightarrow V_3$  defined by  $T(x) = (x, 2x, 3x)$
  - $T: V_1 \rightarrow V_3$  defined by  $T(x) = (x, x^2, x^3)$
  - $T: V_2^C \rightarrow V_3^C$  defined by  $T(x, y) = (x + \alpha, y, 0)$ ,  $\alpha \neq 0$
  - $T: V_2 \rightarrow V_3$  defined by  $T(x, y) = (2x + 3y, 3x - 4y)$
  - $T: V_3 \rightarrow V_3$  defined by  $T(x, y, z) = (x^2 + xy, xy, yz)$
  - $T: V_3 \rightarrow V_3$  defined by  $T(x, y, z) = (x, y)$

- (g)  $T: V_2^{\mathbb{C}} \rightarrow V_2^{\mathbb{C}}$  defined by  $T(x, y) = (y, x)$
- (h)  $T: V_3 \rightarrow V_3$  defined by  $T(x, y, z) = (x + y + z, 0)$
- (i)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p) = p^2 + p$
- (j)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p)(x) = xp(x) + p(1)$
- (k)  $T: \mathcal{C}[0, 1] \rightarrow V_2$  defined by  $T(f) = (f(0), f(1))$
- (l)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p) = p(0)$
- (m)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p)(x) = 2 + 3x + 7x^2 p(x)$
- (n)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p)(x) = p(0) + xp'(0) + \frac{x^2}{2!} p''(0)$
- (o)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p) = p'$
- (p)  $T: \mathcal{C}^{(n)}(a, b) \rightarrow \mathcal{C}^{(n)}(a, b)$  defined by  $T(f) = a_0 f + a_1 f' + \dots + a_n f^{(n)}$ ,  $a_i$ 's are fixed scalars
- (q)  $T: \mathcal{C}^{(2)}(a, b) \rightarrow \mathcal{C}^{(2)}(a, b)$  defined by  $T(f) = (3x^2 + 4)f'' + (7x + 3)f' + (3x + 5)f$ .
3. Determine whether there exists a linear map in the following cases, and where it does exist give the general formula.
- (a)  $T: V_2 \rightarrow V_2$  such that  $T(1, 2) = (3, 0)$  and  $T(2, 1) = (1, 2)$
- (b)  $T: V_2 \rightarrow V_2$  such that  $T(2, 1) = (2, 1)$  and  $T(1, 2) = (4, 2)$
- (c)  $T: V_2 \rightarrow V_2$  such that  $T(0, 1) = (3, 4)$ ,  $T(3, 1) = (2, 2)$ , and  $T(3, 2) = (5, 7)$
- (d)  $T: V_3 \rightarrow V_3$  such that  $T(0, 1, 2) = (3, 1, 2)$  and  $T(1, 1, 1) = (2, 2, 2)$
- (e)  $T: \mathcal{P}_3 \rightarrow \mathcal{P}_3$  such that  $T(1 + x) = 1 + x$ ,  $T(2 + x) = x + 3x^2$ , and  $T(x^2) = 0$
- (f)  $T: \mathcal{P}_4 \rightarrow \mathcal{P}_3$  such that  $T(1 + x) = 1$ ,  $T(x) = 3$ , and  $T(x^2) = 4$
- (g)  $T: V_2^{\mathbb{C}} \rightarrow V_2^{\mathbb{C}}$  such that  $T(i, i) = (1 + i, 1)$ .
4. Determine a nonzero linear transformation  $T: V_2 \rightarrow V_2$ , which maps all the vectors on the line  $x = y$  onto the origin.
5. Determine a linear transformation  $T: V_2 \rightarrow V_2$ , which maps all the vectors on the line  $x + y = 0$  onto themselves ( $T \neq I$ ).
6. Let  $T: V_2^{\mathbb{C}} \rightarrow V_2^{\mathbb{C}}$  be defined by  $T(\alpha_1 + i\beta_1, \alpha_2 + i\beta_2) = (\alpha_1, \alpha_2)$ . Then prove or disprove that  $T$  is linear.
7. Prove that a linear transformation on a 1-dimensional vector space is nothing but multiplication by a fixed scalar.
8. Prove Theorem 4.1.4.
9. True or false?
- (a) There exists a linear transformation  $T: V_2 \rightarrow V_4$  such that  $T(0, 0) = (1, 0, 0, 0)$ .
- (b) Scalar multiplication is the only linear transformation from  $V_1$  to  $V_1$ .
- (c)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined as  $T(p(x)) = xp(x)$  is not a linear transformation.

- (d)  $T: V_3 \rightarrow V_3$  defined by  $T(1, 1) = (1, 0, 0)$ ,  $T(2, 1) = (0, 1, 0)$ ,  $T(0, 1) = (0, 0, 1)$  is not linear.
- (e) Rotation of coordinates in  $V_2$  defined as  $(x, y) \mapsto (x', y')$ , where  $x' = x \cos \theta + y \sin \theta$ ,  $y' = -x \sin \theta + y \cos \theta$ , is a linear transformation.
- (f) Let  $R^+$  be the vector space proved in Problem 2 of Problem Set 3.1. Let  $T: V_3 \rightarrow R^+$  be a linear map. Then  $T(0, 0, 0) = 1$ .
- (g) Linear transformations cannot be defined from the real vector space  $C$  to the complex vector space  $C$ .
- (h) Let  $T: U \rightarrow V$  be a map such that  $T(0_U) \neq 0_V$ . Then  $T$  is not linear.

## 4.2 RANGE AND KERNEL OF A LINEAR MAP

With reference to a linear map  $T: U \rightarrow V$ , two sets are important. One is the range of  $T$ , denoted by  $R(T)$  and already defined (cf Definition 1.4.4) for any function  $T$  as the set of all  $T$ -images. The other is  $N(T)$ , the kernel of  $T$ , defined as follows.

**4.2.1 Definition** Let  $T: U \rightarrow V$  be a linear map. The *kernel (null space)* of  $T$  is the set

$$N(T) = \{u \in U \mid T(u) = 0_V\}.$$

It is also denoted as  $\ker T$ .

In other words,  $N(T)$  is the set of all those elements in  $U$  that are mapped by  $T$  into the zero of  $V$ . Note that this is nothing but the  $T$ -pre-image of  $0_V$  (cf Definition 1.4.2).

As illustration, let us find  $R(T)$  and  $N(T)$  for each of the linear transformations  $T$  defined in Examples 4.1, 4.2, and 4.4-4.8.

**Example 4.11** In Example 4.1 we have  $T: V_3 \rightarrow V_3$  defined by  $T(x_1, x_2, x_3) = (x_1, x_2, 0)$ .

Here  $R(T)$  is the set of all elements of the form  $(x_1, x_2, 0)$ , which is nothing but the  $x_1, x_2$ -plane in  $V_3$ . This also says that  $T$  is not onto (cf Definition 1.4.5).

To determine  $N(T)$ , the kernel of  $T$ , we want all those vectors  $(x_1, x_2, x_3)$  for which  $T(x_1, x_2, x_3) = 0$ . This means  $(x_1, x_2, 0) = (0, 0, 0)$ . So  $x_1 = x_2 = 0$ . In fact, any element of the form  $(0, 0, x_3)$  would be mapped by  $T$  into  $(0, 0, 0)$ . No other element would be so mapped. Therefore,  $N(T)$  is the set of all elements of the form  $(0, 0, x_3)$ , which is nothing but the  $x_3$ -axis in  $V_3$ .

**Example 4.12** In Example 4.2  $T: V_3 \rightarrow V_3$  is defined by  $T(x_1, x_2, x_3) = (x_1 - x_3, x_1 + x_3)$ .

In this case  $R(T)$  consists of vectors of the form  $(x_1 - x_3, x_1 + x_3)$ . We want to determine the vectors of  $V_3$  that are of this form. For this, take

a vector  $(a, b)$  in  $V_2$  and solve the equation

$$(x_1 - x_2, x_1 + x_3) = (a, b).$$

This means  $x_1 - x_2 = a$  and  $x_1 + x_3 = b$ . Solving these, we get

$$x_2 = x_1 - a, x_3 = b - x_1.$$

Hence,  $T(x_1, x_1 - a, b - x_1) = (a, b)$ . This shows that every vector  $(a, b)$  of  $V_2$  is in  $R(T)$ . In other words,  $R(T) = V_2$ . So this is an onto map.

To determine the kernel, we solve the equation  $T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3) = (0, 0)$ . This gives  $x_1 = x_2 = -x_3$ , i.e. all vectors of the form  $(x_1, x_1, -x_1)$  will be mapped into zero. So

$$N(T) = \{x_1(1, 1, -1) \mid x_1 \text{ any scalar}\} = [(1, 1, -1)].$$

This is the subspace of  $V_3$  generated by  $(1, 1, -1)$ .

**Example 4.13** For the zero map (see Example 4.4) the range is  $\{0_V\}$  and the kernel is  $U$ . This is clearly not an onto map.

**Example 4.14** For the identity map  $I_U$  (see Example 4.5) the kernel is  $\{0_U\}$ . The range is  $U$ , so the map is onto.

**Example 4.15** In Example 4.6  $T: V_2 \rightarrow V_2$  is defined by  $T(x_1, x_2) = (x_1, -x_2)$ . Here  $R(T) = V_2$  and  $N(T) = \{(0, 0)\}$ . This is an onto map.

**Example 4.16** In Example 4.7  $D: \mathcal{C}^{(1)}(a, b) \rightarrow \mathcal{C}(a, b)$  is defined by  $D(f) = f'$ . In this case  $R(D) = \mathcal{C}(a, b)$ , since every continuous function  $g$  on  $(a, b)$  possesses an antiderivative and hence  $D$  is an onto map.  $N(D)$  is the set of all constant functions in  $\mathcal{C}^{(1)}(a, b)$ .

**Example 4.17** In Example 4.8  $\mathcal{J}: \mathcal{C}'(a, b) \rightarrow R$  is defined by  $\mathcal{J}(f) = \int_a^b f(x)dx$ . Here the range is the whole of  $R$ , since every real number can be obtained as the algebraic area under some curve  $y = f(x)$  from  $a$  to  $b$ . Therefore, it is an onto map. The kernel is the set of all those functions  $f$  for which the area under the curve  $y = f(x)$  from  $a$  to  $b$  is zero. It is difficult to say anything more than this about the kernel.

We shall now check whether the linear transformations discussed in Examples 4.11 to 4.17 are one-one (cf 1.4.7 for the definition of one-one).

In Example 4.11  $N(T)$  is the  $x_3$ -axis. So all points on the  $x_3$ -axis go into  $(0, 0, 0)$ . So this map is not one-one.

In Example 4.12  $N(T) = [(1, 1, -1)]$ . So, many points go into  $(0, 0, 0)$ . This again means  $T$  is not one-one.

In Example 4.13  $N(T) = U$ . So the zero map is not one-one, because all elements go into the zero of  $V$ .

In Example 4.14 the identity map is one-one, because, if  $x \neq y$ , then certainly  $I(x) \neq I(y)$ . It may be noted that  $N(I)$  is the zero subspace of  $U$ .

In Example 4.15 the linear map  $T$  is one-one, because if  $(x_1, x_2) \neq (y_1, y_2)$ , then  $(x_1, -x_2)$  is also not equal to  $(y_1, -y_2)$ . Observe that in this case also  $N(T)$  is the zero subspace of  $V_2$ .

In Example 4.16, since different functions (say those that differ by a constant) have the same derivative, the map  $D$  is not one one. Observe that  $N(T)$  is a nontrivial subspace in this case. In Example 4.17 the linear map  $T$  is not one-one. (Why?)

Summarising these observations, it appears that  $T$  is one-one when  $N(T)$  is the zero subspace and conversely. This is, in fact, true as a general statement, as is borne out by the following theorem which gives, in addition, more information about  $R(T)$  and  $N(T)$

**4.2.2 Theorem** Let  $T: U \rightarrow V$  be a linear map. Then

- (a)  $R(T)$  is a subspace of  $V$
- (b)  $N(T)$  is a subspace of  $U$ .
- (c)  $T$  is one-one iff  $N(T)$  is the zero subspace,  $\{0_U\}$ , of  $U$ .
- (d) If  $\{u_1, u_2, \dots, u_n\} = U$ , then  $R(T) = [T(u_1), T(u_2), \dots, T(u_n)]$ .
- (e) If  $U$  is finite-dimensional, then  $\dim R(T) \leq \dim U$ .

*Proof:* (a) Let  $v_1, v_2 \in R(T)$ . Then there exist vectors  $u_1, u_2$  in  $U$  such that  $T(u_1) = v_1$  and  $T(u_2) = v_2$ . So

$$v_1 + v_2 = T(u_1) + T(u_2) = T(u_1 + u_2),$$

since  $T$  is linear. But  $u_1 + u_2 \in U$ , since  $U$  is a vector space. Hence,  $v_1 + v_2$  is the image of an element of  $U$ . So  $v_1 + v_2 \in R(T)$ . In the same way,  $\alpha v_1 = \alpha T(u_1) = T(\alpha u_1)$ , since  $T$  is linear. But  $\alpha u_1 \in U$ , because  $U$  is a vector space. Hence,  $\alpha v_1 \in R(T)$ . Thus,  $R(T)$  is a subspace of  $V$ .

(b) Let  $u_1, u_2 \in N(T)$ . Then  $T(u_1) = 0_V$  and  $T(u_2) = 0_V$ , because this is precisely the meaning of their being in  $N(T)$ . Now

$$\begin{aligned} T(u_1 + u_2) &= T(u_1) + T(u_2), \text{ since } T \text{ is linear} \\ &= 0_V + 0_V = 0_V, \end{aligned}$$

which shows that  $u_1 + u_2 \in N(T)$ . Similarly, for all scalars  $\alpha$ , we have

$$\begin{aligned} T(\alpha u_1) &= \alpha T(u_1), \text{ since } T \text{ is linear} \\ &= \alpha 0_V = 0_V \quad (\text{Theorem 3.1.7}) \end{aligned}$$

which shows that  $\alpha u_1 \in N(T)$ . Thus,  $N(T)$  is a subspace of  $U$ .

(c) Suppose  $T$  is one-one. Then  $T(u) = T(v)$  implies  $u = v$ . If  $u \in N(T)$ , then  $T(u) = 0_V = T(0_U)$ . Therefore,  $u = 0_U$ . This means no nonzero vector  $u$  of  $U$  can belong to  $N(T)$ . Since  $0_U$  in any case belongs to  $N(T)$  (why?), it follows that  $N(T)$  contains only  $0_U$  and nothing else. Hence,  $N(T)$  is the zero subspace of  $U$ .

Conversely, suppose  $N(T) = \{0_U\}$ . Then, to prove that  $T$  is one-one, we have to prove that  $T(u) = T(v)$  implies  $u = v$ . Suppose  $T(u) = T(v)$ . Then

$$T(u - v) = T(u) - T(v) = 0_V.$$

So  $u - v \in N(T) = \{0_U\}$ . So  $u - v = 0_U$ , i.e.  $u = v$ . This proves that  $T$  is one-one.

(d) Let  $[u_1, u_2, \dots, u_n] = U$ . Then each vector  $u$  can be expressed as a linear combination of vectors  $u_1, u_2, \dots, u_n$ . The vectors  $T(u_1), T(u_2), \dots, T(u_n)$  are in  $R(T)$ . So, obviously,  $[T(u_1), T(u_2), \dots, T(u_n)] \subset R(T)$ . Let  $v \in R(T)$ . Then there exists a vector  $u \in U$  such that  $T(u) = v$ . Since  $u \in U = [u_1, u_2, \dots, u_n]$ , we have

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n.$$

$$\begin{aligned} \text{Therefore, } v = T(u) &= T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) \\ &= \alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n). \end{aligned}$$

So  $v \in [T(u_1), \dots, T(u_n)]$ . This proves that

$$R(T) = [T(u_1), T(u_2), \dots, T(u_n)].$$

(e) The proof is left to the reader. ■

We shall conclude this article with a definition.

**4.2.3 Definition** Let  $T: U \rightarrow V$  be a linear map. Then

(a) If  $R(T)$  is finite-dimensional, the dimension of  $R(T)$  is called the *rank* of  $T$  and is denoted by  $r(T)$ .

(b) If  $N(T)$  is finite-dimensional, the dimension of  $N(T)$  is called the *nullity* of  $T$  and is denoted by  $n(T)$ .

We shall study these concepts at length in § 4.3.

### Problem Set 4.2

1. Determine the range of the following linear transformations. Also find the rank of  $T$ , where it exists.

(a)  $T: V_2 \rightarrow V_2$  defined by  $T(x_1, x_2) = (x_1 + x_2, x_1)$

(b)  $T: V_2 \rightarrow V_3$  defined by  $T(x_1, x_2) = (x_1, x_1 + x_2, x_2)$

(c)  $T: V_3 \rightarrow V_3$  defined by  $T(x_1, x_2, x_3) = (\frac{1}{2}x_1 + x_2 + x_3, x_1 - \frac{1}{2}x_2, x_3)$

(d)  $T: V_3 \rightarrow V_3$  defined by  $T(x_1, x_2, x_3) = (x_1, x_3, x_2)$

(e)  $T: V_4 \rightarrow V_3$  defined by  $T(x_1, x_2, x_3, x_4) = (x_1 - x_4, x_2 + x_3, x_3 - x_4)$

(f)  $T: V_3 \rightarrow V_4$  defined by  $T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_3 + x_2, x_3)$

(g)  $T: V_4 \rightarrow V_4$  defined by  $T(x_1, x_2, x_3, x_4) = (3x_1 + 2x_2, x_1 - x_3, \frac{1}{2}x_1 - x_4, x_2)$

(h)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p)(x) = xp(x)$

(i)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p)(x) = xp'(x)$

(j)  $T: \mathcal{P} \rightarrow \mathcal{P}$  defined by  $T(p)(x) = p''(x) - 2p(x)$

(k)  $T: \mathcal{C}(0, 1) \rightarrow \mathcal{C}(0, 1)$  defined by  $T(f)(x) = f(x) \sin x$

(l)  $T: \mathcal{C}^{(1)}(0, 1) \rightarrow \mathcal{C}(0, 1)$  defined by  $T(f)(x) = f'(x)e^x$ .

2. Determine the kernel of the linear transformations of Problem 1, (a)-(l). Also find the nullity of  $T$ , where it exists.



3. Let  $T: V \rightarrow W$  be a linear map and  $U$  a subspace of  $V$ . Define  $T(U) = \{w \in W \mid w = T(u) \text{ for some } u \in U\}$ . Then prove that  $T(U)$  is a subspace of  $W$ .
4. Let  $T: V \rightarrow W$  be a linear map and  $W_1$  a subspace of  $W$ . Then prove that the set  $\{v \in V \mid T(v) \in W_1\}$  is a subspace of  $V$ .
5. Let  $\alpha_{11}, \alpha_{12}, \dots, \alpha_{1n}, \alpha_{21}, \dots, \alpha_{2n}, \dots, \alpha_{p1}, \dots, \alpha_{pn}$  be any  $pn$  fixed scalars. Let  $T: V_n \rightarrow V_p$  be a linear map defined by
 
$$T(e_i) = (\alpha_{1i}, \alpha_{2i}, \dots, \alpha_{pi}), \quad i = 1, 2, \dots, n.$$

Then prove that

- (a)  $T$  is not one-one if  $p < n$
- (b)  $T$  is onto when  $p = n$  and  $(\alpha_{11}, \dots, \alpha_{p1}), (\alpha_{12}, \dots, \alpha_{p2}), \dots, (\alpha_{1p}, \dots, \alpha_{pp})$  are LI.
6. Find a linear transformation  $T: V_3 \rightarrow V_3$  such that the set of all vectors  $(x_1, x_2, x_3)$  satisfying the equation  $4x_1 - 3x_2 + x_3 = 0$  is the kernel of  $T$ .
7. Find a linear transformation  $T: V_3 \rightarrow V_3$  such that the set of all vectors  $(x_1, x_2, x_3)$  satisfying the equation  $4x_1 - 3x_2 + x_3 = 0$  is the range of  $T$ .
8. Pick out the maps in Problem 1 that are
  - (a) one-one    (b) onto    (c) one-one and onto.
9. True or false ?
  - (a) Every constant map from one vector space to another is both one-one and onto.
  - (b) A linear transformation  $T: V_3 \rightarrow V_4$  defined as  $T(1, 1) = (1, 0, 0, 0)$  and  $T(1, 2) = (2, 0, 0, 0)$  is one-one.
  - (c) Every linear transformation from  $V_3$  to itself is onto.
  - (d) No linear transformation from  $V_2$  to  $V_3$  is onto.
  - (e) There exist one-one linear transformations from  $V_3$  to  $V_2$ .
  - (f) If  $T: U \rightarrow V$  is a linear transformation and  $v_0 \in V$ , then the  $T$ -pre-image of  $v_0$  is a subspace of  $U$ .

### 4.3 RANK AND NULLITY

A careful scrutiny of Table 4.1 shows that the image of a linearly independent set by a linear map need not be LI. The following theorem throws light on this situation. Essentially, it says that a one-one linear map will preserve linear independence, whereas under *any* linear map the set of pre-images of a linearly independent set of vectors is LI.

**4.3.1 Theorem** Let  $T: U \rightarrow V$  be a linear map. Then

- (a) If  $T$  is one-one and  $u_1, u_2, \dots, u_n$  are linearly independent vectors of  $U$ , then  $T(u_1), T(u_2), \dots, T(u_n)$  are LI.
- (b) If  $v_1, v_2, \dots, v_n$  are linearly independent vectors of  $R(T)$  and

$u_1, u_2, \dots, u_n$  are vectors of  $U$  such that  $T(u_1) = v_1, T(u_2) = v_2, \dots, T(u_n) = v_n$ , then  $u_1, u_2, \dots, u_n$  are LI.

*Proof:* (a) Let  $T$  be one-one and  $u_1, u_2, \dots, u_n$  be linearly independent vectors in  $U$ . To prove that  $T(u_1), T(u_2), \dots, T(u_n)$  are LI, we assume

$$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n) = 0$$

or

$$T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) = 0,$$

since  $T$  is linear. So  $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$ , since  $T$  is one-one. But  $u_1, u_2, \dots, u_n$  are LI. So  $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_n$ .

Thus,  $\alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n) = 0$  implies that each  $\alpha_i$  is zero. Hence,  $T(u_1), T(u_2), \dots, T(u_n)$  are LI.

(b) Let  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  be as stated in Theorem 4.3.1. To prove that  $u_1, u_2, \dots, u_n$  are LI, suppose

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0.$$

Since  $T$  is linear, we have

$$0_V = T(0_U) = T(\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n)$$

or

$$\alpha_1 T(u_1) + \alpha_2 T(u_2) + \dots + \alpha_n T(u_n) = 0_V$$

or

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0_V.$$

But  $v_1, v_2, \dots, v_n$  are LI. Therefore,  $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_n$ .

Thus,  $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0$  implies that  $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_n$ . Hence,  $u_1, u_2, \dots, u_n$  are LI. ■

**Example 4.18** Prove that the linear map  $T: V_3 \rightarrow V_3$  defined by  $T(e_1) = e_1 - e_2, T(e_2) = 2e_2 + e_3, T(e_3) = e_1 + e_2 + e_3$  is neither one-one nor onto.

Since  $[e_1, e_2, e_3] = V_3$ , by Theorem 4.2.2 (d),

$$\begin{aligned} R(T) &= [T(e_1), T(e_2), T(e_3)] = [e_1 - e_2, 2e_2 + e_3, e_1 + e_2 + e_3] \\ &= [e_1 - e_2, 2e_2 + e_3], \end{aligned}$$

because  $e_1 + e_2 + e_3$  is a linear combination of  $e_1 - e_2$  and  $2e_2 + e_3$ . Now we see that  $e_1 - e_2$  and  $2e_2 + e_3$  are LI. So  $\dim R(T) = 2$ . Therefore,  $R(T)$  is a proper subset of  $V_3$ . Hence,  $T$  is not onto.

To prove that  $T$  is not one-one, we check  $N(T)$ .  $N(T)$  consists of those vectors  $(x_1, x_2, x_3)$  in  $V_3$  for which

$$T(x_1, x_2, x_3) = 0$$

or

$$T(x_1 e_1 + x_2 e_2 + x_3 e_3) = 0$$

or

$$x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3) = 0,$$

because  $T$  is linear. Thus,

$$(x_1 + x_3, -x_1 + 2x_2 + x_3, x_3 + x_3) = (0, 0, 0),$$

i.e.  $x_1 + x_3 = 0, x_2 + x_3 = 0$ , and  $-x_1 + 2x_2 + x_3 = 0$ . Solving these, we get  $x_1 = x_2 = -x_3$ . Therefore,

$$N(T) = \{(x_1, x_1, -x_1) \mid x_1 \text{ an arbitrary scalar}\} = \{(1, 1, -1)\}.$$

Hence, by Theorem 4.2.2,  $T$  is not one-one.

In Example 4.18 the linearly independent vectors  $e_1, e_2$ , and  $e_3$  span the domain space  $V_3$ , but their images  $e_1 - e_2, 2e_2 + e_3$ , and  $e_1 + e_2 + e_3$  are LD and  $\dim R(T) = 2$ .

Thus, we find that the effect of the linear map  $T$  on  $U = V_3$  is to shrink  $V_3$  to a 2-dimensional subspace  $R(T)$  of  $V = V_3$ . What happens to the remaining 'one' dimension?

Observe, in this case, that the kernel of  $T$  is  $[(1, 1, -1)]$  and so  $\dim N(T) = 1$ . Thus, it appears that in this example

$$\dim R(T) + \dim N(T) = \dim V_3.$$

This is not an accident. A general result of the same kind is true for all linear maps whose domain space is finite-dimensional. We shall now record this as a major theorem.

**4.3.2 Theorem (Rank-Nullity Theorem)** *Let  $T: U \rightarrow V$  be a linear map and  $U$  a finite-dimensional vector space. Then*

$$\dim R(T) + \dim N(T) = \dim U. \quad (1)$$

*In other words,*

$$r(T) + n(T) = \dim U \quad (2)$$

*or rank + nullity = dimension of the domain space.*

*Proof:*  $N(T)$  is a subspace of a finite-dimensional vector space  $U$ . Therefore,  $N(T)$  is itself a finite-dimensional vector space. Let  $\dim N(T) = n(T) = n$  and  $\dim U = p$  ( $p \geq n$ ). Let  $B = \{u_1, u_2, \dots, u_n\}$  be a basis for  $N(T)$ . Since  $u_i \in N(T)$ ,  $T(u_i) = 0$  for each  $i = 1, 2, \dots, n$ .  $B$  is LI in  $N(T)$  and therefore in  $U$ . Extend this linearly independent set of  $U$  to a basis for  $U$ . Let  $B_1 = \{u_1, u_2, \dots, u_n, u_{n+1}, \dots, u_p\}$  be a basis for  $U$ .

Consider the set

$$A = \{T(u_{n+1}), T(u_{n+2}), \dots, T(u_p)\}.$$

We shall now prove that  $A$  is a basis for  $R(T)$ . Observe that, if this is proved, the proof of the theorem is over; for, this means

$$\dim R(T) = p - n = \dim U - \dim N(T),$$

which is the same as Equation (1).

It is therefore enough to prove

- (i)  $[A] = R(T)$ , and
- (ii)  $A$  is LI.

To prove (i) we proceed as follows: Since  $[B_1] = U$ , it follows from Theorem 4.2.2 (d) that  $R(T) = [T(u_1), T(u_2), \dots, T(u_n), T(u_{n+1}), \dots, T(u_p)]$ . But  $T(u_i) = 0$  for  $i = 1, 2, \dots, n$ . Hence,

$$R(T) = [T(u_{n+1}), T(u_{n+2}), \dots, T(u_p)].$$

To prove (ii), consider

$$\alpha_{n+1}T(u_{n+1}) + \dots + \alpha_p T(u_p) = 0. \quad (3)$$

Using the fact that  $T$  is linear, we get

$$T(\alpha_{n+1}u_{n+1} + \dots + \alpha_p u_p) = 0,$$

which means that  $\alpha_{n+1}u_{n+1} + \dots + \alpha_p u_p \in N(T)$ . Therefore,  $\alpha_{n+1}u_{n+1} + \dots + \alpha_p u_p$  is a unique linear combination of the basis  $B$  for  $N(T)$ . Thus,

$$\alpha_{n+1}u_{n+1} + \dots + \alpha_p u_p = \beta_1 u_1 + \dots + \beta_n u_n, \\ \text{i.e.} \quad \beta_1 u_1 + \dots + \beta_n u_n - \alpha_{n+1}u_{n+1} - \dots - \alpha_p u_p = 0.$$

$B_1$  being a basis for  $U$  is LI. Therefore,

$$\beta_1 = \beta_2 = \dots = \beta_n = \alpha_{n+1} = \dots = \alpha_p = 0.$$

Thus, Equation (3) implies  $\alpha_{n+1} = \dots = \alpha_p = 0$ . Hence,  $A$  is LI. ■

**Example 4.19** Let  $T: V_4 \rightarrow V_3$  be a linear map defined by  $T(e_1) = (1, 1, 1)$ ,  $T(e_2) = (1, -1, 1)$ ,  $T(e_3) = (1, 0, 0)$ ,  $T(e_4) = (1, 0, 1)$ . Then verify that  $r(T) + n(T) = \dim U (= V_4) = 4$ .

We know that  $R(T) = [(1, 1, 1), (1, -1, 1), (1, 0, 0), (1, 0, 1)]$ .  $(1, 1, 1)$ ,  $(1, -1, 1)$ ,  $(1, 0, 0)$ , and  $(1, 0, 1)$  are LD, because a set of four vectors of  $V_3$  ( $\dim V_3 = 3$ ) is always LD. We find that

$$(1, 0, 1) = \frac{1}{2}(1, 1, 1) + \frac{1}{2}(1, -1, 1) + 0(1, 0, 0).$$

Hence, we can discard the vector  $(1, 0, 1)$ , so that

$$R(T) = [(1, 1, 1), (1, -1, 1), (1, 0, 0)].$$

To check whether  $(1, 1, 1)$ ,  $(1, -1, 1)$ , and  $(1, 0, 0)$  are LI, we suppose

$$\alpha_1(1, 1, 1) + \alpha_2(1, -1, 1) + \alpha_3(1, 0, 0) = 0 = (0, 0, 0)$$

$$\text{or} \quad (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 - \alpha_2, \alpha_1 + \alpha_2) = (0, 0, 0).$$

Solving this, we get  $\alpha_1 = 0 = \alpha_2 = \alpha_3$ . Hence,  $(1, 1, 1)$ ,  $(1, -1, 1)$ , and  $(1, 0, 0)$  are LI and  $\dim R(T) = r(T) = 3$ .

Now to find  $N(T)$ , we suppose that  $T(u) = 0 = (0, 0, 0)$ . If

$$u = (x_1, x_2, x_3, x_4) = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4,$$

$$\text{then} \quad T(x_1, x_2, x_3, x_4) = T(x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4) = (0, 0, 0)$$

$$\text{or} \quad (x_1 + x_2 + x_3 + x_4, x_1 - x_2, x_1 + x_2 + x_4) = (0, 0, 0).$$

Solving this, we get  $x_1 = x_2 = -x_4/2$ ,  $x_3 = 0$ . So  $N(T)$  contains the vectors of the form  $(x_1, x_1, 0, -2x_1)$ , i.e.  $N(T) = [(1, 1, 0, -2)]$ . So  $n(T) = \dim N(T) = 1$ . Hence,  $r(T) + n(T) = 3 + 1 = 4$ , and the theorem is verified.

### Problem Set 4.3

1. Let  $U$  be a vector space of dimension  $n$  and  $T: U \rightarrow V$  be a linear and onto map. Then prove that  $T$  is one-one iff  $\dim V = n$ .
2. If  $T: U \rightarrow V$  is a linear map, where  $U$  is finite-dimensional, prove that
  - (a)  $n(T) < \dim U$
  - (b)  $r(T) < \min(\dim U, \dim V)$ .
3. Let  $Z$  be a subspace of a finite-dimensional vector space  $U$ , and  $V$  a finite-dimensional vector space. Then prove that  $Z$  will be the kernel of a linear map  $T: U \rightarrow V$  iff  $\dim Z \geq \dim U - \dim V$ .

4. Prove Theorem 4.3.2 by the following method: Assume a basis  $\{v_1, v_2, \dots, v_r\}$  for  $R(T)$  and  $u_1, u_2, \dots, u_r$  in  $U$  such that  $T(u_i) = v_i$ ,  $i = 1, 2, \dots, r$ . Assume a basis  $\{w_1, w_2, \dots, w_n\}$  for  $N(T)$ . Then prove that  $\{u_1, \dots, u_r, w_1, \dots, w_n\}$  is a basis for  $U$ .
5. True or false?
- No linear transformation from  $V_4$  to  $V_3$  can be one-one.
  - If  $T: V_3 \rightarrow V_3$  is linear and one-one, then it is onto.
  - Let  $T: V_3 \rightarrow V_7$  be a linear map. Then  $R(T)$  can be a 4-dimensional subspace of  $V_7$ .
  - Let  $T: U \rightarrow V$  be a linear map ( $U$  and  $V$  are finite-dimensional vector spaces). If  $T$  is one-one, then  $\dim U < \dim V$ .
  - Polynomial functions  $p$  of degree less than or equal to 3 such that  $\int_a^b p(x)dx = 0$  form a 3-dimensional subspace of  $\mathcal{P}_3$ .

## 4.4 INVERSE OF A LINEAR TRANSFORMATION

Linear transformations that are both one-one and onto play an important role. We give them a special name in the following definition.

**4.4.1 Definition** A linear map  $T: U \rightarrow V$  is said to be *nonsingular* if it is one-one and onto. Such a map is also called an *isomorphism*.

We know that any function has an inverse *iff* it is one-one and onto (cf Definition 1.7.8). Hence, we have the following fact.

**4.4.2 Fact** A linear transformation is nonsingular *iff* it has an inverse.

Let us illustrate this by the following examples.

**Example 4.20** Consider the map  $T: \mathcal{P}_3 \rightarrow V_3$  defined by  $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$ . Clearly,  $T$  is a linear map (check!). This is onto, because given a vector  $(\beta_1, \beta_2, \beta_3)$  in  $V_3$  we can get a polynomial  $\beta_1 + \beta_2 x + \beta_3 x^2$  of which  $(\beta_1, \beta_2, \beta_3)$  is the  $T$ -image.

Further,  $T$  is one-one, because if  $\beta_1 = 0 = \beta_2 = \beta_3$ , then the polynomial  $\beta_1 + \beta_2 x + \beta_3 x^2$  also reduces to the zero polynomial of  $\mathcal{P}_3$ . Thus,  $T$  is an isomorphism. It is easily seen that  $T^{-1}: V_3 \rightarrow \mathcal{P}_3$  is defined as

$$T^{-1}(\alpha_1, \alpha_2, \alpha_3) = \alpha_1 + \alpha_2 x + \alpha_3 x^2.$$

**Example 4.21** In Example 4.15 we have  $T: V_2 \rightarrow V_2$  defined by  $T(x_1, x_2) = (x_1, -x_2)$ . We have already seen that  $T$  is one-one and onto.

To calculate  $T^{-1}(y_1, y_2)$ , we have to find the element that maps into  $(y_1, y_2)$  by  $T$ . The answer is  $(y_1, -y_2)$ , because

$$T(y_1, -y_2) = (y_1, -(-y_2)) = (y_1, y_2).$$

Thus,  $T^{-1}: V_2 \rightarrow V_2$  is defined by  $T^{-1}(y_1, y_2) = (y_1, -y_2)$ .

**Example 4.22** We have already seen that the identity map  $I_U: U \rightarrow U$  defined by  $I_U(u) = u$  for all  $u \in U$  is one-one and onto. So the inverse

of  $I$ , i.e.  $I_U^{-1}$ , exists as a map from  $U$  to  $U$ . Obviously,  $I_U^{-1}(u) = u$  for all  $u \in U$ . Thus,  $I_U^{-1} = I_U$ .

In Examples 4.20-4.22 we can check that  $T^{-1}$  is linear. In fact, this is in general true as proved in the following theorem.

**4.4.3 Theorem** *Let  $T: U \rightarrow V$  be a nonsingular linear map. Then  $T^{-1}: V \rightarrow U$  is a linear, one-one, and onto map.*

*Proof:* To prove the linearity of  $T^{-1}$ , let  $v_1, v_2 \in V$ . Let  $T^{-1}(v_1) = u_1$  and  $T^{-1}(v_2) = u_2$ . Since  $T$  is one-one and onto,  $u_1$  and  $u_2$  exist uniquely. Thus,  $v_1 = T(u_1)$  and  $v_2 = T(u_2)$ . So

$$v_1 + v_2 = T(u_1) + T(u_2) = T(u_1 + u_2),$$

since  $T$  is linear. Therefore,

$$T^{-1}(v_1 + v_2) = u_1 + u_2 = T^{-1}(v_1) + T^{-1}(v_2).$$

Again,  $\alpha v_1 = \alpha T(u_1) = T(\alpha u_1)$ . So

$$T^{-1}(\alpha v_1) = \alpha u_1 = \alpha T^{-1}(v_1).$$

Hence,  $T^{-1}$  is linear.

$T^{-1}$  is onto because, if  $u \in U$ , then  $T(u) = v$  belongs to  $V$ , and  $T^{-1}(v) = u$ .

The rest of the proof is left to the reader. ■

The linear map  $T: V_3 \rightarrow V_3$ , defined in Example 4.18, does not have an inverse, because it is neither one-one nor onto.

A linear transformation  $T: U \rightarrow V$  has an inverse, if the following two properties hold:

- (i)  $T$  is one-one.
- (ii)  $T$  is onto.

If one or both of the properties fail to exist, then  $T^{-1}$  does not exist.

In order to check whether  $T$  is one-one, we have to find  $N(T)$ . If  $N(T)$  is the zero subspace, then  $T$  is one-one, otherwise it is not. The second property, namely,  $T$  is onto, holds iff  $R(T) = V$ . This involves the determination of  $R(T)$ , the range space of  $T$ .

**4.4.4 Remark** Instead of calculating the kernel, if we can somehow find (perhaps by guesswork) more than one vector in  $U$ , which maps into the same vector in  $V$  by  $T$ , then this would be enough to prove that  $T^{-1}$  does not exist. In the case of Example 4.18  $T(1, 1, 0) = (1, 1, 1)$  and  $T(2, 2, -1) = (1, 1, 1)$ .

Just from this we could have concluded that  $T$  does not have an inverse. However, the method that involves finding the kernel is recommended to the reader, since it is applicable in most cases.

We shall now consider two more examples in which  $T^{-1}$  does not exist, because in one case  $T$  is one-one but not onto, and in the other case  $T$  is onto but not one-one.

**Example 4.23** Let  $U$  be the set of all infinite sequences  $\{x_1, x_2, \dots, x_n, \dots\}$  of real numbers. Define the operations of addition and scalar multiplication coordinatewise as in  $V_n$ , i.e. if

$$x = \{x_1, x_2, \dots, x_n, \dots\}$$

and  $y = \{y_1, y_2, \dots, y_n, \dots\},$

then  $x + y = \{x_1 + y_1, x_2 + y_2, \dots, x_n + y_n, \dots\}$

and  $\alpha x = \{\alpha x_1, \alpha x_2, \dots, \alpha x_n, \dots\}.$

The set  $U$  with these two operations becomes a real vector space (check !). Note that the sequence  $\{0, 0, \dots, 0, \dots\}$  is the zero of  $U$  and  $\{-x_1, -x_2, \dots, -x_n, \dots\}$  is the negative of  $x$ .

Let  $T: U \rightarrow U$  be defined by

$$\begin{aligned} T(x) &= T(\{x_1, x_2, \dots, x_n, \dots\}) \\ &= \{x_2, x_3, \dots, x_n, \dots\}. \end{aligned}$$

It is easy to check that  $T$  is linear. Here  $R(T) = U$ . For, take the sequence  $\{y_1, y_2, \dots, y_n, \dots\}$ . Its pre-image by  $T$  can be any sequence of the form  $\{z, y_1, y_2, \dots, y_n, \dots\}$ , where  $z$  can be a real number. In particular,  $z$  can be zero. Hence,  $T$  is onto.

But  $T$  is not one-one, because all the sequences  $\{z, y_1, y_2, \dots, y_n, \dots\}$  map into  $\{y_1, y_2, \dots, y_n, \dots\}$ . Further,  $N(T)$  is the set of all sequences of the form  $\{z, 0, 0, \dots, 0, \dots\}$ .

Thus, though  $T$  is onto it does not have an inverse, since  $T$  is not one-one.

**Example 4.24** Let  $U$  be the vector space of Example 4.23. And let  $T: U \rightarrow U$  be defined by

$$T(\{x_1, x_2, \dots, x_n, \dots\}) = \{0, x_1, x_2, \dots, x_n, \dots\}.$$

Obviously,  $T$  is linear (check !). Now  $T$  is one-one, for  $N(T)$  is the set of all sequences of the form  $\{0, 0, 0, \dots, 0, \dots\}$ . There is only one such sequence, namely, the zero element of the space  $U$ . So  $N(T) = \{0_U\}$ .

But  $T$  is not onto, because the element  $\{1, 1, 1, \dots, 1, \dots\}$  has no pre-image in  $U$ . So  $R(T) \neq U$ . Hence,  $T$  does not have an inverse.

In Examples 4.23 and 4.24 we produced situations where just one of the two conditions, namely, (i)  $T$  is one-one, (ii)  $T$  is onto, holds and the other does not hold. Further, note that in these examples the space  $U$  involved is not finite-dimensional. (Why ?) When  $U$  is finite-dimensional, we may not be able to produce such an example, because of the following theorem, which essentially says that if  $T: U \rightarrow V$  is a linear map and  $\dim U = \dim V$ , then the two conditions (i)  $T$  is one-one and (ii)  $T$  is onto are implications of each other.

**4.4.5 Theorem** *If  $U$  and  $V$  are finite-dimensional vector spaces of the same dimension, then a linear map  $T: U \rightarrow V$  is one-one iff it is onto.*

**Proof:**  $T$  is one-one  $\Leftrightarrow N(T) = \{0\}$  (Theorem 4.2.2)  
 $\Leftrightarrow n(T) = 0$  (Definition 4.2.3)  
 $\Leftrightarrow r(T) = \dim U = \dim V$  (Theorem 4.3.2)  
 $\Leftrightarrow R(T) = V$  (Theorem 3.6.13)  
 $\Leftrightarrow T$  is onto. ■

Now we shall take up an example where the inverse exists and we proceed to calculate it. Recall that a linear map is completely determined as soon as its values on the elements of a basis are specified. Therefore, in order to determine the linear map  $T^{-1}$ , it is enough to determine the values of  $T^{-1}$  on the elements of a basis.

**Example 4.25** Prove that the linear map  $T: V_3 \rightarrow V_3$  defined by  $T(e_1) = e_1 + e_2$ ,  $T(e_2) = e_2 + e_3$ ,  $T(e_3) = e_1 + e_2 + e_3$  is nonsingular, and find its inverse.

First, let us find the value of  $T$  at a general element  $u = (x_1, x_2, x_3)$ :

$$\begin{aligned} T(x_1, x_2, x_3) &= T(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= (x_1 + x_3, x_1 + x_2 + x_3, x_2 + x_3). \end{aligned}$$

If  $T(x_1, x_2, x_3) = 0$ , then

$$x_1 + x_3 = 0, \quad x_1 + x_2 + x_3 = 0, \quad x_2 + x_3 = 0.$$

Solving these, we get  $x_1 = 0 = x_2 = x_3$ . So  $N(T) = \{0_{V_3}\}$  and hence  $T$  is one-one. It follows from Theorem 4.4.5 that  $T$  is also onto. Hence,  $T$  is nonsingular and  $T^{-1}$  exists.

Now we shall give two methods to find the inverse,  $T^{-1}$ , which is also a linear, one-one, and onto map from  $V_3$  to  $V_3$ .

**Method 1** We have  $T(e_1) = e_1 + e_2$ ,  $T(e_2) = e_2 + e_3$ ,  $T(e_3) = e_1 + e_2 + e_3$ . Therefore,

$$\begin{aligned} e_1 &= T^{-1}(e_1 + e_2) = T^{-1}(e_1) + T^{-1}(e_2) \\ e_2 &= T^{-1}(e_2 + e_3) = T^{-1}(e_2) + T^{-1}(e_3) \\ e_3 &= T^{-1}(e_1 + e_2 + e_3) \\ &= T^{-1}(e_1) + T^{-1}(e_2) + T^{-1}(e_3), \end{aligned}$$

because  $T^{-1}$  is linear, one-one, and onto. Solving these three equations for  $T^{-1}(e_1)$ ,  $T^{-1}(e_2)$ , and  $T^{-1}(e_3)$ , we get

$$\begin{aligned} T^{-1}(e_1) &= e_3 - e_2 = (0, -1, 1) \\ T^{-1}(e_2) &= e_1 + e_2 - e_3 = (1, 1, -1) \\ T^{-1}(e_3) &= e_3 - e_1 = (-1, 0, 1). \end{aligned}$$

Now we extend  $T^{-1}$  linearly and obtain

$$\begin{aligned} T^{-1}(x_1, x_2, x_3) &= T^{-1}(x_1 e_1 + x_2 e_2 + x_3 e_3) \\ &= x_1 T^{-1}(e_1) + x_2 T^{-1}(e_2) + x_3 T^{-1}(e_3) \\ &= (x_1 - x_3, x_2 - x_1, x_1 - x_2 + x_3). \end{aligned}$$

**Method 2** Let  $T^{-1}(x_1, x_2, x_3) = (y_1, y_2, y_3)$ . Then

$$T(y_1, y_2, y_3) = (x_1, x_2, x_3)$$



$$\text{or } T(y_1 e_1 + y_2 e_2 + y_3 e_3) = (x_1, x_2, x_3)$$

$$\text{or } y_1 T(e_1) + y_2 T(e_2) + y_3 T(e_3) = (x_1, x_2, x_3)$$

$$\text{or } (y_1 + y_3)e_1 + (y_1 + y_2 + y_3)e_2 + (y_2 + y_3)e_3 = (x_1, x_2, x_3)$$

$$\text{or } (y_1 + y_3, y_1 + y_2 + y_3, y_2 + y_3) = (x_1, x_2, x_3).$$

This gives  $y_1 + y_3 = x_1$ ,  $y_1 + y_2 + y_3 = x_2$ , and  $y_2 + y_3 = x_3$ . Solving these, we get  $y_1 = x_2 - x_3$ ,  $y_2 = x_3 - x_1$ , and  $y_3 = x_1 - x_2 + x_3$ . So

$$T^{-1}(x_1, x_2, x_3) = (x_2 - x_3, x_3 - x_1, x_1 - x_2 + x_3).$$

### Problem Set 4.4

1. Let  $R$ ,  $S$ , and  $T$  be three linear maps from  $V_3$  to  $V_3$  defined by the values in Table 4.2. Determine which of them are nonsingular and in each such case find the inverse.

TABLE 4.2

Value at $\rightarrow$			
Linear maps	$e_1$	$e_2$	$e_3$
$\downarrow$			
$R$	$e_1 + e_2$	$e_1 - e_2 + e_3$	$e_1 + 4e_3$
$S$	$e_1 - e_2$	$e_2$	$e_1 - e_2 - 7e_3$
$T$	$e_1 - e_2 + e_3$	$3e_1 - 5e_3$	$3e_1 - 2e_3$

2. Show that each of the following maps is nonsingular and find its inverse:
  - (a)  $T: V_2 \rightarrow V_2$  defined by  $T(x_1, x_2) = (\alpha_1 x_1, \alpha_2 x_2)$ , where  $\alpha_1$  and  $\alpha_2$  are both nonzero.
  - (b)  $T: V_3 \rightarrow V_3$  defined by  $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_3 + x_1, x_3)$ .
  - (c)  $T: \mathcal{P}_2 \rightarrow \mathcal{P}_2$  defined by  $T(\sigma_0 + \alpha_1 x + \alpha_2 x^2) = (\sigma_0 + \alpha_1 + (\alpha_1 + 2\alpha_2)x + (\sigma_0 + \alpha_1 + 2\alpha_2)x^2)$ .
3. Let  $U$  be the subset  $\{p \in \mathcal{P} \mid p(0) = 0\}$  of  $\mathcal{P}$ . Then prove that the derivative  $D$  is a nonsingular linear map from  $U$  to  $\mathcal{P}$ , and the integral  $(\mathcal{I}(p))(x) = \int_0^x p(x)dx$  is its inverse.
4. Let  $T: U \rightarrow V$  be a nonsingular linear transformation. Then prove that  $(T^{-1})^{-1} = T$ .
5. True or false?
  - (a) Every linear map from  $V_3$  to  $V_3$  has an inverse.
  - (b) The inverse of a nonsingular linear map is nonsingular.
  - (c) A nonsingular linear map transforms linearly independent sets into linearly independent sets.

- (d) A nonsingular linear map transforms linearly dependent sets into linearly dependent sets.
- (e) Every translation of  $V_2$  to  $V_2$  has an inverse.
- (f) Every translation of  $V_2$  to  $V_2$  is an isomorphism.
- (g) Given two vectors  $u_1, u_2 \in U$ , there exists an isomorphism  $T: U \rightarrow U$  such that  $T(u_1) = u_2$ .

## 4.5 CONSEQUENCES OF RANK-NULLITY THEOREM

The results discussed in § 4.4 and certain allied facts are important for further discussions. We collect them together in the following theorem.

**4.5.1 Theorem** *Let  $T: U \rightarrow V$  be a linear map and  $\dim U = \dim V = p$ . Then the following statements are equivalent :*

- (a)  $T$  is nonsingular (an isomorphism).
- (b)  $T$  is one-one.
- (c)  $T$  transforms linearly independent subsets of  $U$  into linearly independent subsets of  $V$ .
- (d)  $T$  transforms every basis for  $U$  into a basis for  $V$ .
- (e)  $T$  is onto.
- (f)  $r(T) = p$ .
- (g)  $n(T) = 0$ .
- (h)  $T^{-1}$  exists.

*Proof:* (a)  $\Rightarrow$  (b) by Definition 4.4.1.

(b)  $\Rightarrow$  (c) by Theorem 4.3.1(a).

(c)  $\Rightarrow$  (d). Let  $T$  transform linearly independent subsets of  $U$  into linearly independent subsets of  $R(T)$ . Now, let  $\{u_1, u_2, \dots, u_p\}$  be a basis for  $U$ . Then  $T(u_1), T(u_2), \dots, T(u_p)$  are LI by hypothesis. But  $\dim V = p$ . So  $\{T(u_1), T(u_2), \dots, T(u_p)\}$  is a basis for  $V$  by Theorem 3.6.7.

(d)  $\Rightarrow$  (e). Let  $\{u_1, u_2, \dots, u_p\}$  be a basis for  $U$ . Then, by hypothesis,  $\{T(u_1), T(u_2), \dots, T(u_p)\}$  is a basis for  $V$ . This means, by Theorem 4.2.2 (d),  $R(T) = V$ . Hence,  $T$  is onto.

(e)  $\Rightarrow$  (f).  $T$  is onto  $\Rightarrow R(T) = V \Rightarrow r(T) = p$ .

(f)  $\Rightarrow$  (g) by the Rank-Nullity theorem.

(g)  $\Rightarrow$  (h).  $n(T) = 0$  means  $N(T) = \{0_U\}$ , i.e.  $T$  is one-one. So  $T$  is onto by Theorem 4.4.4. Hence,  $T^{-1}$  exists.

(h)  $\Rightarrow$  (a) by Fact 4.4.2.  $\square$

We shall conclude this article by exhibiting an important isomorphism of linear algebra.

**4.5.2 Definition** Two vector spaces  $U$  and  $V$  are said to be *isomorphic* if there exists an isomorphism from  $U$  to  $V$ . If  $U$  and  $V$  are isomorphic, then we write  $U \cong V$ .

**4.5.3 Theorem** Every real (complex) vector space of dimension  $p$  is isomorphic to  $V_p$  ( $V_p^C$ ).

*Proof:* Let  $U$  be a real vector space of dimension  $p$ ,  $B = \{u_1, u_2, \dots, u_p\}$  an ordered basis for  $U$ ,  $u$  an arbitrary element of  $U$ , and  $(\alpha_1, \alpha_2, \dots, \alpha_p)$  the coordinate vector of  $u$  relative to  $B$ . Consider the map  $T : U \rightarrow V_p$  defined by

$$T(u) = (\alpha_1, \alpha_2, \dots, \alpha_p).$$

This map is easily seen to be linear; for, if

$$\begin{aligned} u &= \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_p u_p \\ \text{and} \quad v &= \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_p u_p, \\ \text{then} \quad u + v &= (\alpha_1 + \beta_1)u_1 + (\alpha_2 + \beta_2)u_2 + \dots + (\alpha_p + \beta_p)u_p. \end{aligned}$$

Hence, the coordinate vector of  $u + v$  relative to  $B$  is  $(\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_p + \beta_p)$ . So

$$T(u + v) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_p + \beta_p).$$

$$\text{But} \quad T(u) + T(v) = (\alpha_1, \alpha_2, \dots, \alpha_p) + (\beta_1, \beta_2, \dots, \beta_p).$$

$$\text{Therefore,} \quad T(u + v) = T(u) + T(v).$$

$$\text{Similarly,} \quad T(\alpha u) = \alpha T(u).$$

Further,  $T$  is one-one, because  $T(u) = (\alpha_1, \alpha_2, \dots, \alpha_p) = 0_V$  means  $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_p$ , which implies  $u = 0u_1 + 0u_2 + \dots + 0u_p = 0_U$ , i.e.  $N(T) = \{0_U\}$ . Thus, by Theorem 4.5.1,  $T$  is an isomorphism. Hence, there exists an isomorphism from  $U$  to  $V_p$  and consequently  $U \simeq V_p$ . The proof of the other part is left to the reader. ■

**Example 4.26** In Example 4.20 we have seen that  $T : \mathcal{P}_2 \rightarrow V_3$  defined by  $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0, \alpha_1, \alpha_2)$  is an isomorphism. Thus,  $\mathcal{P}_2 \simeq V_3$ .

Theorem 4.5.3 says that not only  $\mathcal{P}_2$  but also any real vector space of dimension 3 is isomorphic to  $V_3$ .

### Problem Set 4.5

1. Prove that ' $\simeq$ ' is an equivalence relation.

2. Let  $A$  be the subspace of  $V_4$  defined by

$$A = \{(x_1, x_2, x_3, x_4) \mid x_2 = 0\}.$$

Prove, by exhibiting an isomorphism, that  $A \simeq V_3$ .

3. Let  $B$  be the subspace of  $\mathcal{P}_4$  defined by

$$B = \{p \mid p'(1) = 0, p''(1) = 0\}.$$

Prove, by exhibiting an isomorphism, that  $B \simeq V_3$ .

4. Prove, by exhibiting an isomorphism, that  $A \simeq B$ , where  $A$  and  $B$  are subspaces of Problems 2 and 3.

5. True or false ?

(a) The differential operator  $D : \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}$  has nullity zero.

(b) There exist two isomorphisms from  $\mathcal{P}_3$  to  $V_3$ .

- (c) In  $V_2$  all nontrivial subspaces are isomorphic.
- (d) Rotation in  $V_2$  is an isomorphism.
- (e) Let  $T: V \rightarrow V$  ( $V$  is a finite-dimensional vector space) be a linear map. If  $R(T) \cap N(T) = \{0_V\}$ , then  $V = R(T) \oplus N(T)$ .

## 4.6 THE SPACE $L(U, V)$

### SUM OF TWO LINEAR MAPS

Let  $T: U \rightarrow V$  and  $S: U \rightarrow V$  be two linear transformations. Consider the map  $M: U \rightarrow V$  defined by

$$M(u) = S(u) + T(u) \quad \text{for all } u \in U.$$

We shall prove that  $M$  is linear, that is,

$$M(u_1 + u_2) = M(u_1) + M(u_2)$$

and

$$M(\alpha u_1) = \alpha M(u_1),$$

for all  $u_1, u_2 \in U$  and all scalars  $\alpha$ .

We have

$$\begin{aligned} M(u_1 + u_2) &= S(u_1 + u_2) + T(u_1 + u_2) && \text{(definition of } M) \\ &= (S(u_1) + S(u_2)) + (T(u_1) + T(u_2)), \end{aligned}$$

because  $S$  and  $T$  are linear maps. On the other hand,

$$\begin{aligned} M(u_1) + M(u_2) &= (S(u_1) + T(u_1)) + (S(u_2) + T(u_2)) && \text{(definition of } M) \\ &= (S(u_1) + S(u_2)) + (T(u_1) + T(u_2)) \end{aligned}$$

by associativity and commutativity of addition in  $V$ . Thus,

$$M(u_1 + u_2) = M(u_1) + M(u_2) \quad \text{for all } u_1, u_2 \in U.$$

Again,

$$\begin{aligned} M(\alpha u_1) &= S(\alpha u_1) + T(\alpha u_1) && \text{(definition of } M) \\ &= \alpha(S(u_1)) + \alpha(T(u_1)) && (S, T \text{ are linear}) \\ &= \alpha(S(u_1) + T(u_1)) \end{aligned}$$

by properties of scalar multiplication in  $V$ . Thus,  $M(\alpha u_1) = \alpha M(u_1)$  for all  $u_1 \in U$  and all scalars  $\alpha$ . This proves that  $M$  is a linear map.

The map  $M$  defined above is called the *sum of  $S$  and  $T$*  and is denoted by  $S + T$ . Thus,

$$(S + T)(u) = M(u) = S(u) + T(u) \quad \text{for all } u \in U. \quad (1)$$

We have thus shown that the sum of two linear maps is linear.

### SCALAR MULTIPLE OF A LINEAR MAP

Let  $S: U \rightarrow V$  be a linear map and  $\alpha$  a given scalar. Note that  $U$  and  $V$  are vector spaces over the same field of scalars, and  $\alpha$  also belongs to the same field. Consider the map  $P: U \rightarrow V$  defined by

$$P(u) = \alpha(S(u)) \quad \text{for all } u \in U.$$

We shall prove that  $P$  is linear.

Let  $u_1, u_2$  be two elements of  $U$  and  $\lambda$  be a scalar. Then

$$\begin{aligned} P(u_1 + u_2) &= \alpha(S(u_1 + u_2)) && \text{(definition of } P) \\ &= \alpha(S(u_1) + S(u_2)) && (S \text{ is linear}) \\ &= \alpha(S(u_1)) + \alpha(S(u_2)) && (V \text{ is a vector space}) \\ &= P(u_1) + P(u_2) && \text{(definition of } P). \end{aligned}$$

Again,

$$\begin{aligned} P(\lambda u_1) &= \alpha(S(\lambda u_1)) && \text{(definition of } P) \\ &= \alpha(\lambda(S(u_1))) && (S \text{ is linear}) \\ &= \lambda(\alpha(S(u_1))) && (V \text{ is a vector space}) \\ &= \lambda(P(u_1)) && \text{for each scalar } \lambda \text{ (definition of } P). \end{aligned}$$

This proves that  $P$  is a linear map.

The map  $P$  defined above is called the *scalar multiple of  $S$  by  $\alpha$*  and is denoted by  $\alpha S$ . Thus,

$$(\alpha S)(u) = P(u) = \alpha(S(u)) \quad \text{for all } u \in U. \quad (2)$$

Thus, the scalar multiple of a linear map is linear.

**Example 4.27** Let  $T: V_3 \rightarrow V_2$  and  $S: V_3 \rightarrow V_2$  be two linear maps defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, x_2 + x_3)$$

and

$$S(x_1, x_2, x_3) = (2x_1, x_2 - x_3).$$

Then  $(S + T): V_3 \rightarrow V_2$  is given by

$$\begin{aligned} (S + T)(x_1, x_2, x_3) &= S(x_1, x_2, x_3) + T(x_1, x_2, x_3) \\ &= (x_1 - x_2, x_2 + x_3) + (2x_1, x_2 - x_3) \\ &= (3x_1 - x_2, 2x_2); \end{aligned}$$

and  $\alpha S: V_3 \rightarrow V_2$  is given by

$$\begin{aligned} (\alpha S)(x_1, x_2, x_3) &= \alpha(S(x_1, x_2, x_3)) \\ &= \alpha(x_1 - x_2, x_2 + x_3) \\ &= (\alpha x_1 - \alpha x_2, \alpha x_2 + \alpha x_3). \end{aligned}$$

**Example 4.28** Let  $T: V_3 \rightarrow V_3$  and  $S: V_3 \rightarrow V_3$  be two linear maps defined by

$$T(e_1) = e_1 + e_2, \quad T(e_2) = e_3, \quad T(e_3) = e_2 - e_3;$$

and

$$S(e_1) = e_3, \quad S(e_2) = 2e_2 - e_3, \quad S(e_3) = 0.$$

Then  $S + T: V_3 \rightarrow V_3$  is given by

$$\begin{aligned} (S + T)(e_1) &= S(e_1) + T(e_1) = e_1 + e_2 + e_3, \\ (S + T)(e_2) &= S(e_2) + T(e_2) = 2e_2, \\ (S + T)(e_3) &= S(e_3) + T(e_3) = e_2 - e_3; \end{aligned}$$

and  $2T: V_3 \rightarrow V_3$  is given by

$$\begin{aligned} (2T)(e_1) &= 2(T(e_1)) = 2e_1 + 2e_2, \\ (2T)(e_2) &= 2(T(e_2)) = 2e_3, \\ (2T)(e_3) &= 2(T(e_3)) = 2e_2 - 2e_3. \end{aligned}$$

The set of *all* linear transformations from  $U$  to  $V$  is denoted by  $L(U, V)$ . We now have three different objects before us (see Figure 4.2), namely,

- (i) the vector space  $U$ ,
- (ii) the vector space  $V$ , and
- (iii) the set  $L(U, V)$ , whose elements are linear maps  $T : U \rightarrow V$ .

The sum of two linear maps and the scalar multiple of a linear map of the foregoing discussion give us the operations of addition and scalar multiplication in  $L(U, V)$ . Our claim is that  $L(U, V)$  is a vector space for this addition and scalar multiplication.

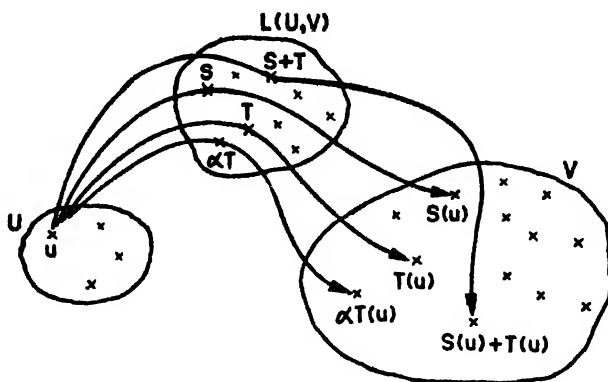


FIGURE 4.2

Since the sum of two linear transformations is a linear transformation and the scalar multiple of a linear transformation is linear, we have only to check axiom (VS3). First note that  $L(U, V)$  is a commutative group for addition. The zero map  $0 : U \rightarrow V$ , defined by  $0(u) = 0_V$  for all  $u \in U$ , is in  $L(U, V)$  (see Example 4.4) and plays the role of identity for addition in  $L(U, V)$ . The map  $(-S) : U \rightarrow V$  is defined by  $(-S)(u) = -(S(u))$  for all  $u \in U$ . It is easily seen that  $(-S) \in L(U, V)$  and  $-S = (-1)S$ .

Further, if  $S$  and  $T$  are two members of  $L(U, V)$  and  $\alpha, \beta$  two scalars, then

- (i)  $\alpha(S + T) = \alpha S + \alpha T$ ,
- (ii)  $(\alpha + \beta)S = \alpha S + \beta S$ ,
- (iii)  $\alpha(\beta S) = (\alpha\beta)S = \beta(\alpha S)$ ,
- (iv)  $1S = S$ .

We shall leave it to the reader to verify these axioms. Once they are verified, we will have proved the following theorem:

**4.6.1 Theorem** *The set  $L(U, V)$  of all linear transformations from  $U$  to  $V$  together with the operations of addition and scalar multiplication defined in Statements (i) and (ii) is a vector space.*

**4.6.2 Remark** It should be noted that  $L(U, V)$  is a real vector space if both  $U$  and  $V$  are real vector spaces, and it is a complex vector space

if both  $U$  and  $V$  are complex vector spaces.

We shall prove in Chapter 5 that if  $U$  and  $V$  are finite-dimensional vector spaces, then  $\dim L(U, V) = \dim U \times \dim V$ .

### Problem Set 4.6

1. Let the linear maps  $T: V_2 \rightarrow V_2$  and  $S: V_2 \rightarrow V_2$  be defined by

$$\begin{aligned} T(x_1, x_2) &= (x_1 + x_2, 0), \\ S(x_1, x_2) &= (2x_1, 3x_1 + 4x_2). \end{aligned}$$

Determine the linear maps

(a)  $2S + 3T$       (b)  $3S - 7T$ .

2. Let the linear maps  $T: V_3 \rightarrow V_3$  and  $S: V_3 \rightarrow V_3$  be defined by

$$\begin{aligned} T(x_1, x_2, x_3) &= (2x_1 - 3x_2, 4x_1 + 6x_2, x_3) \\ S(e_1) &= e_2 - e_3, S(e_2) = e_1, S(e_3) = e_1 + e_2 + e_3. \end{aligned}$$

Determine the linear maps

(a)  $S + T$       (b)  $3S - 2T$       (c)  $\alpha S$

and find their values at  $(x_1, x_2, x_3)$ .

3. Let the linear maps  $R, S$ , and  $T$  be defined as in Problem 1, § 4.4. Then find the linear maps

(a)  $R + 2S$       (b)  $2R + 5T$       (c)  $S - T$   
(d)  $R + S + 2T$       (e)  $\alpha R + \beta S + \gamma T$ .

4. Prove that the set of all linear maps from  $V_2$  to  $V_2$  which map the vectors on the line  $x + y = 0$  onto the origin is a subspace of  $L(V_2, V_2)$ .
5. Let  $U$  be a subspace of a vector space  $V$ . Then prove that the set of all linear transformations from  $V$  to  $V$  that vanish on  $U$  is a subspace of  $L(V, V)$ .
6. Let  $T: V_1 \rightarrow V_1$  be a nonzero linear transformation. Then prove that  $L(V_1, V_1) = [T]$ .
7. Determine two linear transformations  $T$  and  $S$  of rank 4 from  $V_4$  to  $V_4$  such that
- (a)  $r(T + S) = 3$       (b)  $r(T - S) = 2$   
(c)  $r(T + 2S) = 1$       (d)  $r(T - S) = 0$ .
8. Prove Theorem 4.6.1.
9. If  $S$  and  $T$  belong to  $L(U, U)$ , then prove that  $S \circ T \in L(U, U)$ .
10. True or false?
- (a) To prove  $\alpha(S + T) = \alpha S + \alpha T$ , where  $S$  and  $T$  are linear maps from  $U$  to  $V$ , it is enough to prove that  $S + T$  is linear.
- (b)  $L(U, R)$  is a proper subset of  $\mathcal{F}_R(U)$ .
- (c) If  $S, T: U \rightarrow V$  are linear, then  $S - T$  is linear.

- (d) For a linear map  $S$ ,  $-S(u) = S(-u)$ .
- (e)  $I_U \in L(U, V)$ .
- (f) Rank of  $(S + T) = \text{rank of } S + \text{rank of } T$ .
- (g) Rank of  $(\alpha S) = \alpha \text{ rank of } S$ .

## 4.7 COMPOSITION OF LINEAR MAPS

Let  $T: U \rightarrow V$  and  $S: V \rightarrow W$  be two linear maps. We know that the composition  $S \circ T: U \rightarrow W$  is defined by

$$S \circ T(u) = S(T(u)) \quad \text{for all } u \in U \quad (\text{Definition 1.7.2}).$$

$$\text{Symbolically, } U \xrightarrow{S \circ T} W = U \xrightarrow{T} V \xrightarrow{S} W.$$

This map is linear, because with the usual terminology

$$\begin{aligned} S \circ T(u_1 + u_2) &= S(T(u_1 + u_2)) && (\text{definition of composition}) \\ &= S(T(u_1) + T(u_2)) && (T \text{ is linear}) \\ &= S(T(u_1)) + S(T(u_2)) && (S \text{ is linear}) \\ &= (S \circ T)(u_1) + (S \circ T)(u_2) && (\text{definition of composition}) \end{aligned}$$

for all  $u_1, u_2 \in U$ .

$$\begin{aligned} \text{Again, } (S \circ T)(\alpha u_1) &= S(T(\alpha u_1)) && (\text{definition of composition}) \\ &= S(\alpha(T(u_1))) && (T \text{ is linear}) \\ &= \alpha(S(T(u_1))) && (S \text{ is linear}) \\ &= \alpha(S \circ T)(u_1) && (\text{definition of composition}) \end{aligned}$$

for all  $u_1 \in U$  and all scalars  $\alpha$ .

Thus, the composition of two linear maps is a linear map.

**Example 4.29** Let a linear map  $T: V_3 \rightarrow V_4$  be defined by

$$T(e_1) = (1, 1, 0, 0), T(e_2) = (1, -1, 1, 0), T(e_3) = (0, -1, 1, 1),$$

where  $\{e_1, e_2, e_3\}$  is the standard basis for  $V_3$ , and let a linear map  $S: V_4 \rightarrow V_3$  be defined by

$$S(f_1) = (1, 0), S(f_2) = (1, 1), S(f_3) = (1, -1), S(f_4) = (0, 1),$$

where  $\{f_1, f_2, f_3, f_4\}$  is the standard basis for  $V_4$ . Then the linear map  $S \circ T: V_3 \rightarrow V_3$  is obtained as follows (see Figure 4.3) :

$$\begin{aligned} (S \circ T)(e_1) &= S(T(e_1)) = S(1, 1, 0, 0) = S(f_1 + f_2) \\ &= S(f_1) + S(f_2) = (1, 0) + (1, 1) = (2, 1), \\ (S \circ T)(e_2) &= S(T(e_2)) = S(1, -1, 1, 0) = S(f_1 - f_2 + f_3) \\ &= S(f_1) - S(f_2) + S(f_3) \\ &= (1, 0) - (1, 1) + (1, -1) = (1, -2), \\ (S \circ T)(e_3) &= S(T(e_3)) = S(0, -1, 1, 1) = S(-f_2 + f_3 + f_4) \\ &= -S(f_2) + S(f_3) + S(f_4) \\ &= -(1, 1) + (1, -1) + (0, 1) = (0, -1). \end{aligned}$$



We shall hereafter use  $ST$  for  $S \circ T$  and call it the *product of  $S$  and  $T$* , rather than the composition of  $S$  and  $T$ .

We know that if  $ST$  is defined, then  $TS$  need not be defined. Even if both are defined, they need not be equal. Thus, the commutative law of the product is *not* in general satisfied. The other laws of multiplication are easily seen to hold.

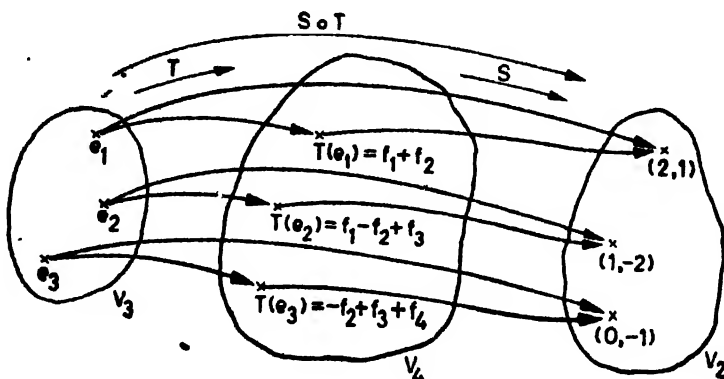


FIGURE 4.3

**4.7.1 Theorem** Let  $T_1, T_2$  be linear maps from  $U$  to  $V$ . Let  $S_1, S_2$  be linear maps from  $V$  to  $W$ . Let  $P$  be a linear map from  $W$  to  $Z$ , where  $U, V, W$ , and  $Z$  are vector spaces over the same field of scalars. Then

- (a)  $S_1(T_1 + T_2) = S_1T_1 + S_1T_2$ .
- (b)  $(S_1 + S_2)T_1 = S_1T_1 + S_2T_1$ .
- (c)  $P(S_1T_1) = (PS_1)T_1$ .
- (d)  $(\alpha S_1)T_1 = \alpha(S_1T_1) = S_1(\alpha T_1)$ , where  $\alpha$  is a scalar.
- (e)  $I_V T_1 = T_1$  and  $T_1 I_U = T_1$ .

*Proof:* (a)  $T_1 + T_2 : U \rightarrow V$  and  $S_1 : V \rightarrow W$  are linear. The products  $S_1(T_1 + T_2)$  and  $S_1T_1, S_1T_2$  are defined. So both sides of (a) make sense. Now, if  $u$  is a vector of  $U$ , then

$$\begin{aligned}
 (S_1(T_1 + T_2))(u) &= S_1((T_1 + T_2)(u)) && \text{(definition of product)} \\
 &= S_1(T_1(u) + T_2(u)) && \text{(definition of addition in } L(U, V)) \\
 &= S_1(T_1(u)) + S_1(T_2(u)) && (S_1 \text{ is linear)} \\
 &= (S_1T_1)(u) + (S_1T_2)(u) && \text{(definition of product)} \\
 &= (S_1T_1 + S_1T_2)(u) && \text{(definition of addition in } L(U, V)).
 \end{aligned}$$

Hence,

$$S_1(T_1 + T_2) = S_1T_1 + S_1T_2.$$

The proofs of the remaining parts are left to the reader. ■

Note specially the diagram for part (e) of Theorem 4.7.1 :

$$\begin{aligned}
 U &\xrightarrow{T_1} V \xrightarrow{I_V} V = U \xrightarrow{T_1} V \\
 U &\xrightarrow{I_U} U \xrightarrow{T_1} V = U \xrightarrow{T_1} V.
 \end{aligned}$$

Let  $T: U \rightarrow V$  be a nonsingular linear map, i.e.  $T$  is linear, one-one, and onto. Then we know that  $T^{-1}: V \rightarrow U$  exists and is linear (Theorem 4.4.3). Further,  $TT^{-1} = I_V$  and  $T^{-1}T = I_U$  (cf § 1.7). In fact, this characterises nonsingularity as shown in the following theorem

**4.7.2 Theorem** *A linear map  $T: U \rightarrow V$  is nonsingular iff there exists a linear map  $S: V \rightarrow U$  such that  $TS = I_V$  and  $ST = I_U$ .*

*In such a case  $S = T^{-1}$  and  $T = S^{-1}$ .*

*Proof:* Let  $T$  be nonsingular. The existence of  $S$ , namely,  $T^{-1}$ , follows from the observations made immediately before the theorem.

Conversely, let  $S$  and  $T$  exist with the properties stated, i.e.  $TS = I_V$  and  $ST = I_U$ .

Let  $u \in N(T)$ . Then  $T(u) = 0$  and consequently  $S(T(u)) = 0$ . But  $S(T(u)) = I_U(u) = u$ . Therefore,  $u = 0$ . So  $N(T)$  is the zero subspace of  $U$ . Hence,  $T$  is one-one.

Now let  $v \in V$ . Then  $T(S(v)) = (TS)(v) = I_V(v) = v$ . Therefore, there exists an element  $u = S(v)$  in  $U$  such that  $T(u) = v$ . Hence,  $T$  is onto. Thus,  $T$  is nonsingular. Therefore,  $T^{-1}$  exists by Fact 4.4.2. Further, using Theorem 4.7.1, we get

$$T^{-1} = T^{-1}(I_V) = T^{-1}(TS) = (T^{-1}T)S = I_US = S.$$

Similarly,  $T = S^{-1}$ . ■

Finally, we prove the following interesting theorem.

**4.7.3 Theorem** *Let  $T: U \rightarrow V$  and  $S: V \rightarrow W$  be two linear maps. Then*

(a) *If  $S$  and  $T$  are nonsingular, then  $ST$  is nonsingular and*

$$(ST)^{-1} = T^{-1}S^{-1}.$$

(b) *If  $ST$  is one-one, then  $T$  is one-one.*

(c) *If  $ST$  is onto, then  $S$  is onto.*

(d) *If  $ST$  is nonsingular, then  $T$  is one-one and  $S$  is onto.*

(e) *If  $U, V, W$  are of the same finite dimension and  $ST$  is nonsingular, then both  $S$  and  $T$  are nonsingular.*

*Proof:* (a) Since  $S$  is nonsingular,  $S^{-1}$  is defined and  $SS^{-1} = I_W$  and  $S^{-1}S = I_V$ .

Since  $T$  is nonsingular,  $T^{-1}$  is defined and  $TT^{-1} = I_V$  and  $T^{-1}T = I_U$ . By virtue of (c) and (e) of Theorem 4.7.1, we have

$$\begin{aligned}
 (ST)(T^{-1}S^{-1}) &= S(T(T^{-1}S^{-1})) = S((TT^{-1})S^{-1}) \\
 &= S(I_V S^{-1}) = SS^{-1} = I_W.
 \end{aligned}$$

Similarly,

$$\begin{aligned}(T^{-1}S^{-1})(ST) &= T^{-1}(S^{-1}(ST)) = T^{-1}((S^{-1}S)T) \\ &= T^{-1}(I_V T) = T^{-1}T = I_U.\end{aligned}$$

Hence, by Theorem 4.7.2,  $ST$  is nonsingular and  $(ST)^{-1} = T^{-1}S^{-1}$ .

(b) Let  $u \in N(T)$ . Then  $T(u) = 0_V$ . So  $S(T(u)) = 0_W$ , i.e.  $(ST)(u) = 0_W$ . But  $ST$  is one-one. Therefore,  $u = 0_U$ , i.e.  $N(T) = \{0_U\}$ . Thus,  $T$  is one-one.

(c) Let  $w \in W$ . Since  $ST$  is onto, there exists a vector  $u \in U$  such that  $(ST)(u) = w$ . Therefore,  $S(T(u)) = w$ . Hence, there exists a vector  $v = T(u) \in V$  such that  $S(v) = w$ . Thus,  $S$  is onto.

The proofs of (d) and (e) are left to the reader.  $\square$

**Notations** (a) The set of all linear operators on  $U$  is denoted by  $L(U)$ .

(b) If  $T$  is a linear transformation on  $U$ , then the composition  $TT$  is also denoted by  $T^2$ . Similarly,  $T^k = T^{k-1}T$  for any positive integer  $k$ . By convention,  $T^0 = I$ .

### Problem Set 4.7

- Let  $T: V_3 \rightarrow V_3$  be defined by  $T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, x_1)$  and  $S: V_2 \rightarrow V_2$  be defined by  $S(x_1, x_2) = (x_2, x_1)$ . Then determine  $ST$ .
- Let  $S$  and  $T$  be as in Problem 1, § 4.6. Determine  
(a)  $ST$  (b)  $TS$  (c)  $S^2$  (d)  $T^2S$ .
- Let  $S$  and  $T$  be as in Problem 2, § 4.6. Determine  
(a)  $ST$  (b)  $TS$  (c)  $STS$  (d)  $TST$ .
- Let  $R, S, T$  be as in Problem 1, § 4.4. Determine  
(a)  $ST$  (b)  $RT$  (c)  $RST$  (d)  $R(S + T)$  (e)  $T^2$   
(f)  $T^2ST$ . Also verify that  $R(S + T) = RS + RT$ .
- Determine two linear transformations  $S$  and  $T$  on  $V_2$  such that  $ST = 0_{L(V_2)}$  and  $TS \neq 0_{L(V_2)}$ .
- Let  $S$  and  $T$  be two linear maps on  $U$  such that  $ST = TS$ . Then prove that  
(a)  $(S + T)^2 = S^2 + 2(ST) + T^2$   
(b)  $(S + T)^n = S^n + nC_1 S^{n-1}T + \dots + nC_n T^n$ .  
(Hint: Use induction.)
- Let  $V$  be a 1-dimensional vector space and  $S, T$  two linear maps on  $V$ . Then prove that  $ST = TS$ .
- Let  $T$  be a linear map on a 1-dimensional vector space  $V$ . Then prove that  $T^2 = \alpha T$  for some fixed scalar  $\alpha$ .
- Find the range, kernel, rank, and nullity of the following linear

maps, where  $R, S, T$  are as in Problem 1 of Problem Set 4.4 :

(a)  $RS$     (b)  $RT$     (c)  $RST$ .

10. Let  $T$  be a linear map on a finite-dimensional vector space  $V$ . Then prove that
  - (a)  $R(T) \cap N(T) = \{0\}$  iff  $T^2x = 0 \Rightarrow Tx = 0$
  - (b) If  $r(T^2) = r(T)$ , then  $R(T) \cap N(T) = \{0\}$ .
11. If a linear transformation  $T$  on  $V$  satisfies the condition  $T^2 + I = T$ , then prove that  $T^{-1}$  exists.
12. Let  $T$  be a linear map on  $V_3$  defined by  $T(e_1) = e_3, T(e_2) = e_1, T(e_3) = e_2$ . Then prove that  $T^3 = I$ .
13. Let  $T: U \rightarrow V$  and  $S: V \rightarrow W$  be two linear maps. Then prove that
  - (a) If  $T$  is onto, then  $r(ST) = r(S)$
  - (b) If  $S$  is one-one, then  $r(ST) = r(T)$ .
14. A linear transformation  $T$  on a vector space  $V$  is said to be *idempotent* if  $T^2 = T$ . For example, the zero transformation and the identity transformation are idempotent.
  - (a) Let  $S$  and  $T$  be two linear maps on  $V_3$  defined as
 
$$T(x_1, x_2, x_3) = (0, x_2, x_3)$$

$$S(x_1, x_2, x_3) = (x_1, 0, 0).$$
 Then prove that both  $S$  and  $T$  are idempotent.
  - (b) If  $S_1$  and  $S_2$  are idempotent on a vector space  $V$ , then find the conditions under which  $S_1S_2$  and  $S_1 + S_2$  are idempotent.
  - (c) If  $S$  is idempotent on a vector space  $V$ , then  $I - S$  is also idempotent on  $V$ .
  - (d) Determine two idempotent transformations  $S$  and  $T$  on a vector space  $V$  such that  $ST = TS = 0$ , but neither  $S = 0$  nor  $T = 0$ .
  - (e) If  $T$  is an idempotent transformation on  $V$ , then prove that
 
$$N(T) = R(I - T) \text{ and } R(T) = N(I - T).$$
15. A linear transformation  $T$  on a vector space  $V$  is said to be *nilpotent* on  $V$  if  $T^n = 0$  for some integer  $n > 1$ , and the smallest such integer ' $n$ ' is called the *degree of nilpotence* of  $T$ .
  - (a) Prove that the differential operator  $D$  is nilpotent on  $\mathcal{P}_n$ . What is the degree of nilpotence of  $D$ ?
  - (b) Is  $D^3 + D$  nilpotent on  $\mathcal{P}_4$ ? If yes, find its degree of nilpotence.
  - (c) Let  $T: V_4 \rightarrow V_4$  be defined by
 
$$T(x_1, x_2, x_3, x_4) = (0, x_1, x_2, x_3).$$
 Is  $T$  nilpotent?

- (d) Let
- $T: V_3 \rightarrow V_3$
- be defined by

$$T(e_1) = 0, T(e_2) = e_1, T(e_3) = 2e_1 + 3e_3.$$

Is  $T$  nilpotent?

- (e) If
- $S$
- and
- $T$
- are nilpotent transformations on a vector space
- $V$
- and
- $ST = TS$
- , then prove that
- $ST$
- is nilpotent on
- $V$
- . What is the degree of nilpotence of
- $ST$
- ?

What can you say about the nilpotence of  $S + T$ ?

16. Let
- $T: V_4 \rightarrow V_4$
- be defined by

$$T(x_1, x_2, x_3, x_4) = (0, 2x_1, 3x_1 + 2x_2, x_2 + 4x_3).$$

Then prove that

- (a)  $T$  is nilpotent of degree 4  
 (b) For a nonzero scalar  $\lambda$ ,  $I + \lambda T$  is nonsingular and  $(I + \lambda T)^{-1} = I - \lambda T + \lambda^2 T^2 - \lambda^3 T^3$ . In particular,  $I + T$  and  $I - T$  are nonsingular.

17. Let
- $T: \mathcal{P}_3 \rightarrow \mathcal{P}_3$
- be defined by

$$T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = (x_0 + 2\alpha_1)x^2 + (\alpha_1 + \alpha_2)x^3.$$

Then prove that

- (a)  $T$  is nilpotent of degree 3  
 (b) For a nonzero scalar  $\lambda$ ,  $I + \lambda T$  is nonsingular. Find its inverse.

18. True or false?

- (a) If
- $S: U \rightarrow V$
- ,
- $T: V \rightarrow W$
- are linear, then

$$(T \circ S)(\alpha u) = ((\alpha T) \circ (\alpha S))(u).$$

- (b)
- $T(S(\alpha u)) = (\alpha T)(S(u))$
- .

- (c) If
- $S$
- is one-one, then
- $TS$
- is one-one.

- (d) Every idempotent operator is nonsingular.

- (e) A nilpotent operator can never be nonsingular.

- (f)
- $T^2 = T \Rightarrow T^{-1} = (T^{-1})^2$
- , if
- $T^{-1}$
- exists.

- (g) If
- $T$
- is idempotent, then
- $T^k = T$
- for all positive integers
- $k$
- .

- (h) If
- $S: U \rightarrow V$
- and
- $T: V \rightarrow U$
- are linear and
- $TS = IU$
- , then
- $S = T^{-1}$
- .

- (i) If
- $R, S, T$
- are three linear transformations such that
- $RST$
- is defined and is one-one, then
- $T$
- is one-one.

## 4.8 OPERATOR EQUATIONS

Let  $T: U \rightarrow V$  be a linear map from the vector space  $U$  to the vector space  $V$ . In this article we shall discuss the solutions of the equation

$$T(u) = v_0, \quad (1)$$

where  $v_0$  is a fixed vector in  $V$ . Equation (1) is called an *operator equation*.

It is said to be *homogeneous* if  $v_0 = 0_V$ . The set of solutions of the equation

$$T(u) = 0_V \quad (2)$$

is simply the kernel of  $T$ , i.e.  $N(T)$ . If the equation is not homogeneous, i.e.  $v_0 \neq 0_V$  in Equation (1), then (1) is called the *nonhomogeneous* (NH) equation and (2) the homogeneous (H) equation associated with (1). In this connection we have the following theorem.

**4.8.1 Theorem** *Let  $T: U \rightarrow V$  be a linear map. Given  $v_0 \neq 0_V$  in  $V$ , the nonhomogeneous equation*

$$(NH) \quad T(u) = v_0$$

*and the associated homogeneous equation*

$$(H) \quad T(u) = 0_V$$

*have the following properties:*

- (a) *If  $v_0 \notin R(T)$ , then (NH) has no solution for  $u$ .*
- (b) *If  $v_0 \in R(T)$  and (H) has the trivial solution, namely,  $u = 0_U$ , as its only solution, then (NH) has a unique solution.*
- (c) *If  $v_0 \in R(T)$  and (H) has a nontrivial solution, namely, a solution  $u \neq 0_U$ , then (NH) has an infinite number of solutions. In this case if  $u_0$  is a solution of (NH), then the set of all solutions of (NH) is the linear variety  $u_0 + K$ , where  $K = N(T)$  is the set of all solutions of (H).*

*Proof:* (a) is obvious. Recall the definition of  $R(T)$ .

(b) If  $v_0 \in R(T)$ , then  $T(u) = v_0$  has a solution. If  $T(u) = 0_V$  has only one solution, i.e.  $u = 0_U$ , then  $N(T) = \{0_U\}$ , i.e.  $T$  is one-one. This means  $T(u) = v_0$  cannot have more than one solution, i.e. the solution of (NH) is unique.

(c) If  $T(u) = 0_V$  has a nonzero solution, then  $N(T) \neq \{0_U\}$ . Let  $u_0 \in U$  be a solution of (NH). It exists because  $v_0 \in R(T)$ . Then  $T(u_0) = v_0$ . Now if  $u_k \in N(T)$ , then

$$\begin{aligned} T(u_0 + u_k) &= T(u_0) + T(u_k) \\ &= v_0 + 0_V = v_0. \end{aligned}$$

Therefore,  $u_0 + u_k$  is a solution of (NH). This is true for every  $u_k \in N(T)$ , and since the latter has an infinite number of elements in it, (NH) also has an infinite number of solutions.

From this discussion it is obvious that  $u_0 + K$ , where  $K = N(T)$ , is contained in the solution set of (NH). Conversely, if  $w$  be any other solution of (NH), then

$$T(w) = v_0 = T(u_0)$$

or

$$T(w - u_0) = 0_V.$$

This means  $w - u_0 \in N(T) = K$ . So  $w$  and  $u_0$  belong to the same

parallel of  $K$ , namely,  $u_0 + K$ . Thus, the solution set of (NH) is precisely  $u_0 + K$ . ■

Note that  $u_0 + K$  is the  $T$ -pre-image of  $v_0$ .

**Example 4.30** Let  $D : \mathcal{C}'^{(1)}(0, 2\pi) \rightarrow \mathcal{C}'(0, 2\pi)$  be the linear differential operator. Consider the operator equation

$$D(f)(x) = \sin x.$$

To solve this, we look at the associated homogeneous equation

$$(H) \quad Df = 0.$$

The solution set of this equation is the set of all constant functions. So

$$K = \{f \mid f(x) = b \text{ for all } x \in (0, 2\pi) \text{ and } b \text{ a constant}\}.$$

One solution of  $D(f)(x) = \sin x$  is the function  $f_0$ , where  $f_0(x) = -\cos x$ . So the solution set is

$$f_0 + K.$$

In other words, the set of all functions  $g$ , where  $g(x) = -\cos x + (\text{a constant})$  is the solution set of  $D(f)(x) = \sin x$ .

To solve a nonhomogeneous operator equation

$$(NH) \quad T(u) = v_0,$$

where  $T$  is a linear operator, we go through three steps :

**Step 1** Form the associated homogeneous equation (H).

**Step 2** Get all solutions of (H). It is the kernel of  $T$ , i.e.  $N(T)$ .

**Step 3** Get one particular solution  $u_0$  of (NH).

Now the complete solution of (NH) is  $u_0 + N(T)$ .

**Example 4.31** Let  $T : V_5 \rightarrow V_3$  be a linear map defined by

$$\begin{aligned} T(e_1) &= \frac{1}{2}f_1, \quad T(e_2) = \frac{1}{2}f_1, \quad T(e_3) = f_2, \\ T(e_4) &= f_3, \quad T(e_5) = 0, \end{aligned}$$

where  $\{e_1, e_2, e_3, e_4, e_5\}$  is the standard basis for  $V_5$  and  $\{f_1, f_2, f_3\}$  is the standard basis for  $V_3$ . Then solve the equation

$$T(u) = (1, 1, 0).$$

We first calculate the value of  $T(u)$ , i.e.  $T(x_1, x_2, x_3, x_4, x_5)$  :

$$\begin{aligned} T(x_1, x_2, x_3, x_4, x_5) &= x_1T(e_1) + x_2T(e_2) + x_3T(e_3) + x_4T(e_4) \\ &+ x_5T(e_5) = \left(\frac{x_1}{2} + \frac{x_2}{2}, x_3 + x_4, 0\right). \end{aligned}$$

The associated homogeneous equation leads to the equations

Solving these, we get  $x_2 = -x_1$ ,  $x_3 = -x_4$ . Thus, the kernel of  $T$  is the set of all vectors of the form  $(x_1, -x_1, x_3, -x_3, x_5)$ , i.e.  $x_1(1, -1, 0, 0, 0) + x_3(0, 0, 1, -1, 0) + x_5(0, 0, 0, 0, 1)$ . Hence,

$$N(T) = [(1, -1, 0, 0, 0), (0, 0, 1, -1, 0), (0, 0, 0, 0, 1)].$$

One particular solution of  $T(u) = (1, 1, 0)$  is  $u_0 = (2, 0, 1, 0, 0)$ , which is obtained by putting  $x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 0$ . So the complete solution of the equation

$$T(u) = (1, 1, 0)$$

is the linear variety  $(2, 0, 1, 0, 0) \vdash N(T)$ , i.e. the set

$$(2, 0, 1, 0, 0) + \{(a, -a, b, -b, c) \mid a, b, c \text{ are real numbers}\},$$

which is the same as

$$\{(a + 2, -a, b + 1, -b, c) \mid a, b, c \text{ are real numbers}\}.$$

In other words, the  $T$ -pre-image of  $(1, 1, 0)$  is this linear variety.

### Problem Set 4.8

1. Determine the range, kernel, and the pre-image of  $(3, -1, 2)$  for the linear transformations  $R, S, T$  of Problem 1, Problem Set 4.4.
2. Find the  $T$ -pre-image of  $(1, 2, 3)$  under the linear transformations  $T$  defined in (b) through (e) of Problem 1 Problem Set 4.2.
3. Find the  $T$ -pre-image of the following two vectors under the linear transformation defined in (h) through (l) of Problem 1, Problem Set 4.2 :  
(a)  $x$       (b)  $x^2$ .

## 4.9 APPLICATIONS TO THE THEORY OF ORDINARY LINEAR DIFFERENTIAL EQUATIONS

In this article we propose to apply the theory of operator equations to the important operator equation in mathematics, namely, the linear ordinary differential equation. For this we shall develop the theory of ordinary linear differential equation, using the necessary concepts from linear algebra.

The simplest linear ordinary differential equation of the first order is of the form

$$a_0(x) \frac{dy}{dx} + a_1(x)y = g(x), \quad (1)$$

where  $a_0(x)$ ,  $a_1(x)$ , and  $g(x)$  are continuous on an interval  $I$ . If  $a_0(x)$  is nowhere zero in  $I$ , then Equation (1) is called a *normal* linear differential equation of first order. If Equation (1) is normal, we can conveniently write it in the form

$$\frac{dy}{dx} + \frac{a_1(x)}{a_0(x)}y = \frac{g(x)}{a_0(x)}. \quad (2)$$

This is usually written as

$$\frac{dy}{dx} + Py = Q, \quad (3)$$



where  $P$  and  $Q$  are continuous functions of  $x$  alone. We can rewrite it in operator form as

$$(D + P)y = Q$$

or

$$Ly = Q, \quad (4)$$

where  $L$  stands for the operator  $D + P$  which is a linear operator from  $\mathcal{C}^{(1)}(I)$  to  $\mathcal{C}(I)$ . Since  $L$  is a linear operator, we can look at Equation (4) as an operator equation. Therefore, we can apply the results of § 4.8. In particular, we know that the general solution of Equation (4) consists of two parts. One part is the solution of the homogeneous equation

$$(H) \quad Ly = 0$$

and the other is a particular solution of

$$(NH) \quad Ly = Q.$$

The general solution of (H), i.e. of

$$\frac{dy}{dx} + Py = 0 \quad (5)$$

is called the *complementary function* and is denoted by  $y_c$ . A particular solution of (NH), i.e. of Equation (3) is called a *particular integral* and is denoted by  $y_p$ .

Therefore, the complete solution of Equation (3) is of the form

$$y = y_c + y_p, \quad (6)$$

where  $y_c$  is actually the kernel of  $L$  and  $y_p$  is some particular solution of Equation (3).

To solve Equation (3), we first find  $y_c$ . We write Equation (5) in the differential form

$$Pydx + dy = 0 \quad (7)$$

or 
$$Pdx + \frac{1}{y} dy = 0.$$

Integrating Equation (7), we get

$$\int Pdx + \ln y = \ln C, \quad (8)$$

where  $C$  is an arbitrary constant. Equation (8) can also be written as

$$y = Ce^{-\int Pdx}, \quad (9)$$

which is the required complementary function  $y_c$ . Writing Equation (9) as  $y e^{\int Pdx} = C$ , we find that it is the solution of

$$\frac{d}{dx} (y e^{\int Pdx}) = 0,$$

i.e. 
$$P y e^{\int Pdx} + e^{\int Pdx} \frac{dy}{dx} = 0.$$

This gives the clue to the solution of Equation (3). We multiply both sides of Equation (3) by  $e^{\int Pdx}$  and get

$$\frac{d}{dx} (y e^{\int Pdx}) = Q e^{\int Pdx},$$

which on integration gives

$$ye^{\int Pdx} = \int Qe^{\int Pdx} dx.$$

We are not interested in any arbitrary constant now, because we are looking for only one  $y_p$ . So

$$y_p = e^{-\int Pdx} \int Qe^{\int Pdx} dx.$$

Thus, the complete solution of Equation (3) is

$$y = y_c + y_p = Ce^{-\int Pdx} + e^{-\int Pdx} \int Qe^{\int Pdx} dx, \quad (10)$$

where  $C$  is an arbitrary constant.

**Example 4.32**  $\frac{dy}{dx} - \frac{xy}{x^2 - 1} = x.$

Here  $P = -\frac{x}{x^2 - 1}$ ,  $Q = x$ , and  $\int Pdx = -\int \frac{x}{x^2 - 1} dx$   
 $= -\frac{1}{2} \ln |x^2 - 1|.$

So  $e^{\int Pdx} = e^{-\frac{1}{2} \ln |x^2 - 1|} = 1/\sqrt{x^2 - 1}$  and  $e^{-\int Pdx} = \sqrt{x^2 - 1}.$

Thus,

$$y_c = C\sqrt{x^2 - 1},$$

$$y_p = \sqrt{x^2 - 1} \int \frac{x}{\sqrt{x^2 - 1}} dx = (x^2 - 1),$$

and the complete solution is  $y = y_c + y_p.$

**Example 4.33**

$$\frac{dy}{dx} + Py = Qy^n, \quad n \neq 1. \quad (11)$$

This equation is the well-known *Bernoulli's equation*. It is not linear, but it can be reduced to the linear form by the substitution

$$Y = y^{1-n}, \quad \frac{dY}{dx} = (1 - n)y^{-n} \frac{dy}{dx}.$$

Equation (11) then becomes

$$\frac{1}{1 - n} \frac{dY}{dx} + PY = Q.$$

This is a linear differential equation and can be solved by the method of Example 4.32.

Equation (10) represents a family of solutions (actually it is a linear variety). For each value of  $C$ , this is a curve in the  $xy$ -plane. If, in addition, we require a solution that satisfies the condition  $y(x_0) = y_0$ , then the solution will be represented by a unique member of the family passing through  $(x_0, y_0)$ . The problem of finding a solution  $y = y(x)$  for Equation (3), which also satisfies the initial condition  $y(x_0) = y_0$ , is called an *initial value problem* for the normal first order linear differential equation.

The foregoing discussion is summarised in the following theorem.

**4.9.1 Theorem** *Every initial value problem involving a normal first order ordinary linear differential equation has one and only one solution.*

This is called an *existence and uniqueness theorem*—existence, because the theorem asserts that every initial value problem has a solution; uniqueness, because it further says that the solution is unique.

Existence and uniqueness theorems are important in the theory of differential equations. Theorem 4.9.1 was easily provable. The corresponding theorem for the  $n$ -th order linear differential equation is beyond the scope of this book. However, we shall now state it (without proof), since we intend to use it in the theory that follows.

#### 4.9.2 Theorem *Let*

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x)y = g(x), \quad (12)$$

$a_0(x) \neq 0$ ,  $x \in I$  be a normal  $n$ -th order linear differential equation defined on an interval  $I$ . Let  $x_0$  be a fixed point in  $I$ . Let  $\alpha_0, \alpha_1, \dots, \alpha_{n-1}$  be  $n$  arbitrary real numbers. Then there exists one and only one solution of Equation (12) satisfying the initial conditions  $y(x_0) = \alpha_0$ ,  $y'(x_0) = \alpha_1, \dots, y^{(n-1)}(x_0) = \alpha_{n-1}$ .

The problem of finding a solution of Equation (12) satisfying the initial conditions, listed in Theorem 4.9.2, is called an *initial value problem* for the equation.

This theorem is at the basis of all further theory of linear differential equations. It says that every initial value problem involving a normal  $n$ -th order linear differential equation has precisely one solution. We are now ready to state the main theorem of this article.

#### 4.9.3 Theorem *The solution space of any normal $n$ -th order homogeneous linear ordinary differential equation*

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x)y = 0, \quad (13)$$

$a_0(x) \neq 0$ ,  $x \in I$ , defined on an interval  $I$  is an  $n$ -dimensional subspace of  $\mathcal{C}^{(n)}(I)$ .

To prove this theorem, we need the following lemmas.

#### 4.9.4 Lemma *The solution space of Equation (13) is a subspace of $\mathcal{C}^{(n)}(I)$ .*

*Proof:* We need only to note that the differential operator  $L = a_0(x)D^n + a_1(x)D^{n-1} + \dots + a_n(x)$  is a linear operator from  $\mathcal{C}^{(n)}(I)$  to  $\mathcal{C}(I)$ , and so the solution space of Equation (13) is the kernel of  $L$ . Therefore, it is a subspace of  $\mathcal{C}^{(n)}(I)$ .

#### 4.9.5 Lemma *Let $y_1, y_2, \dots, y_n$ be $n$ functions in $\mathcal{C}^{(n)}(I)$ and $x_0$ a fixed point in $I$ . If the vectors $v_i(x_0) = (y_i(x_0), y'_i(x_0), \dots, y_i^{(n-1)}(x_0))$ , $i = 1, 2, \dots, n$ of $V_n$ are LI, then $y_1, y_2, \dots, y_n$ are LI over $I$ .*

*Proof:* To prove that  $y_1, y_2, \dots, y_n$  are LI over  $I$ , assume

$$c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) = 0 \quad \text{for all } x \in I. \quad (14)$$



of Equation (13) satisfying the said initial conditions. Hence, by the uniqueness part of Theorem 4.9.2,  $y = \alpha_1 y_1 + \dots + \alpha_n y_n$ . So  $y \in [y_1, y_2, \dots, y_n]$ .

Thus, the set  $\{y_1, y_2, \dots, y_n\}$  is LI and spans  $K$ , which means that the set  $\{y_1, y_2, \dots, y_n\}$  is a basis for  $K$ . Hence,  $\dim K = n$ .

From Theorem 4.9.3, the following corollary is obvious.

**4.9.6 Corollary** *If  $y_1, y_2, \dots, y_n$  are  $n$  linearly independent solutions of Equation (13), then any solution of Equation (13) can be written as  $C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ .*

In Theorem 4.9.3 and Corollary 4.9.6 we have discussed in detail the solution of the homogeneous normal linear differential equation  $Ly = 0$ , where  $L = a_0(x)D^n + a_1(x)D^{n-1} + \dots + a_{n-1}(x)D + a_n(x)$ . We now consider the nonhomogeneous equation

$$Ly = g. \quad (15)$$

Since  $L$  is a linear operator from  $\mathcal{C}^{(n)}(I)$  to  $\mathcal{C}(I)$ , Equation (15) can be considered as an operator equation. Hence, by Theorem 4.8.1, the general solution of Equation (15) is the  $L$ -pre-image of  $g$ . It is the linear variety  $y_p + y_c$ , where  $y_p$  is a particular solution of Equation (15) and  $y_c$  is the kernel of  $L$ . We summarise this result in the form of a theorem.

**4.9.7 Theorem** *Let*

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x)y = g(x) \quad (16)$$

*be a nonhomogeneous normal linear differential equation on an interval  $I$ . Then the set  $S$  of all solutions of Equation (16) is a linear variety  $S = y_p + y_c$  in  $\mathcal{C}^{(n)}(I)$ , where the leader  $y_p$  is any one solution of the equation and  $y_c$  is the set of all solutions of the associated homogeneous equation.*

### Problem Set 4.9

1. Solve the following differential equations :

$$(a) \quad \frac{dy}{dx} - y = xe^{x^2+x} \quad (b) \quad x \frac{dy}{dx} + y = x^2 + x^2 \cos x$$

$$(c) \quad (4-3x) \frac{dy}{dx} - 5y = x^2/(3x-4)^{2/3} \quad (d) \quad \frac{dy}{dx} + \alpha y = e^{\beta x}$$

$$(e) \quad (x + 2y^3) \frac{dy}{dx} = y \quad (f) \quad \frac{dy}{dx} = \alpha x + \beta y + \delta$$

$$(g) \quad (\tan^{-1} y - x) \frac{dy}{dx} = 1 + y^2 \quad (h) \quad \sec y \, dx - (x + \cos y) dy = 0$$

$$(i) \quad (y^3 - 1)dx + (2x - y^3 + 1)dy = 0$$

$$(j) \quad \cos y \, dx + (x \sin y - 1)dy = 0$$

$$(k) \quad (x^2 + 1) \frac{dy}{dx} + 2xy = \sin x + xe^x + x^2$$

$$(l) \quad (x^2 + 1) \frac{dy}{dx} + xy = \frac{1}{x(1 + x^2)}.$$

2. Solve the following differential equations :

$$(a) \quad (1 + x^2) \frac{dy}{dx} = xy - y^2 \quad (b) \quad \frac{dy}{dx} = x^3y^3 - xy$$

$$(c) \quad \frac{dy}{dx} + y^3e^{-x^2} = xy \quad (d) \quad e^y dx = x(2xy + e^y)dy$$

$$(e) \quad \cos y \, dx + x(x - \sin y)dy = 0.$$

3. In each of the following differential equations, verify that the given functions are solutions. Determine whether they are LI and whether they form a basis for the solution space.

$$(a) \quad y'' + 5y' + 6y = 0; e^{-2x}, e^{-3x} \text{ on } (-\infty, \infty).$$

$$(b) \quad y'' - 4y' = 0; e^{2x}, e^{-2x} \text{ on } (-\infty, \infty).$$

$$(c) \quad x^2y'' - 2xy' + 2y = 0; 2x^2 - x, x^2 + 2x, x^2 \text{ on } (-\infty, \infty).$$

$$(d) \quad y''' - y'' - 2y' = 0; e^{-x}, \sinh x - e^x/2, 2e^{2x}, 1 \text{ on } (-\infty, \infty).$$

$$(e) \quad y'' - 2xy'/(1 - x^2) = 0; 1 - \tanh^{-1}x, 1 + \ln \left| \frac{1 - \lambda}{1 + \lambda} \right|, 2 \text{ on } (-1, 1).$$

$$(f) \quad (D^4 - 4D^3 + 7D^2 - 6D + 2)y = 0; xe^x, e^x, e^x(1 - \sin x), e^x \cos x, e^x(\sin x - x) \text{ on } (-\infty, \infty).$$

$$(g) \quad (x^2D^2 + 3xD + 1)y = 0; 1/x, \ln x/x \text{ on } (0, \infty).$$

$$(h) \quad (xD^3 - D^2)y = 0; 1, x, x^3 \text{ on } (0, \infty).$$

## Chapter 5

# Matrices

Matrices form an important tool in the study of finite-dimensional vector spaces. Determinants form an important tool in the study of matrices. We shall study matrices in this chapter and determinants in the next.

Hereafter, though for convenience we deal with real vectors and the real vector space  $V_n$ , all our arguments and definitions, *unless otherwise restricted*, will apply also to complex vectors and the complex vector space  $V_n^{\mathbb{C}}$ .

## 5.1 MATRIX ASSOCIATED WITH A LINEAR MAP

For the theory of matrices, it is sometimes necessary to visualise vectors of  $V_p$  (any  $p$ -dimensional real vector space is isomorphic to  $V_p$ ) as *column vectors*, i.e. a vector  $(\alpha_1, \alpha_2, \dots, \alpha_p)$  of  $V_p$  may be written in the form

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_p \end{bmatrix},$$

where coordinates of the vector are written in a vertical column read from top to bottom. It may be noted that so far we have been writing this as a *row vector*, i.e. in a horizontal row from left to right. To save space, column vectors are also written as  $(\alpha_1, \alpha_2, \dots, \alpha_p)^T$ , where the letter  $T$  denotes that the vector is a column vector. (The appropriateness of the letter  $T$  will become clear later.)

For example, the vectors  $e_1$ ,  $e_2$ , and  $e_3$  of  $V_3$  may be written as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or as  $(1, 0, 0)^T$ ,  $(0, 1, 0)^T$ , and  $(0, 0, 1)^T$ . If  $v = 2e_1 - 3e_2 + e_3$ , then the coordinate vector  $(2, -3, 1)$  of  $v$  relative to the basis  $\{e_1, e_2, e_3\}$  can be written as

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \text{ or } (2, -3, 1)^T.$$

Before we discuss the general definition of a matrix, let us take a simple example: Let  $B_1 = \{u_1, u_2, u_3\}$  and  $B_2 = \{v_1, v_2, v_3, v_4\}$  be ordered bases of  $V_3$  and  $V_4$ , respectively. Let  $T: V_3 \rightarrow V_4$  be a linear map defined by

$$\begin{aligned} T(u_1) &= v_1 - 2v_2 + v_3 - v_4 \\ T(u_2) &= v_1 - v_2 + 2v_4 \\ T(u_3) &= 2v_2 + 3v_3 - v_4. \end{aligned}$$

Then the coordinate vectors (relative to  $B_2$ ) of  $T(u_1)$ ,  $T(u_2)$ , and  $T(u_3)$ , written as column vectors, are respectively

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 2 \\ 3 \\ -1 \end{bmatrix}.$$

Everything about the linear map  $T$  is contained in these  $3 \times 4 = 12$  numbers, written as above. The arrangement of these 12 numbers in the form

$$\begin{bmatrix} 1 & 1 & 0 \\ -2 & -1 & 2 \\ 1 & 0 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

is called the matrix of  $T$  relative to  $B_1$  and  $B_2$ .

We shall now give the general definition of a matrix.

**5.1.1 Definition** Let  $U$  and  $V$  be vector spaces of dimensions  $n$  and  $m$ , respectively. Let  $B_1 = \{u_1, \dots, u_n\}$  and  $B_2 = \{v_1, \dots, v_m\}$  be ordered bases of  $U$  and  $V$ , respectively. Let  $T: U \rightarrow V$  be a linear map defined by  $T(u_j) = \alpha_{1j}v_1 + \alpha_{2j}v_2 + \dots + \alpha_{mj}v_m$ ,  $j = 1, 2, \dots, n$  so that the coordinate vector of  $T(u_j)$  written as a column vector is



$$\begin{bmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{mj} \end{bmatrix}.$$

Write the coordinate vectors of  $T(u_1), T(u_2), \dots, T(u_j), \dots, T(u_n)$  successively as column vectors in the form of a rectangular array as follows :

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1j} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2j} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mj} & \dots & \alpha_{mn} \end{bmatrix}.$$

This rectangular array is called the *matrix of  $T$  relative to the ordered bases  $B_1$  and  $B_2$* , and is denoted by  $(T: B_1, B_2)$ .

It may be noted that in this definition  $\alpha_{ij}$  is the  $i$ th coordinate of  $T(u_j)$  relative to the basis  $\{v_1, v_2, \dots, v_m\}$ .

**5.1.2 Remark** The numbers that constitute a matrix are called its *entries*. Each horizontal line of entries is called a *row*. Each vertical line of entries is called a *column*. The above matrix has  $m$  rows and  $n$  columns. The matrix is therefore called an ' $m \times n$ ' (read as  $m$  by  $n$ ) matrix. The order of the matrix is said to be ' $m \times n$ '. The  $j$ -th column of the matrix is a column vector  $(\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{mj})^T$ , and may be considered an element of  $V_m$ . The  $i$ -th row of the matrix is a row vector  $(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in})$ , and may be considered an element of  $V_n$ . The matrix is denoted, in short, by the symbol  $(\alpha_{ij})_{\substack{i=1, \dots, m \\ j=1, \dots, n}}$  or

simply  $(\alpha_{ij})_{m \times n}$ .  $\alpha_{ij}$  is called the  $ij$ -th entry, and it is at the intersection of the  $i$ -th row and the  $j$ -th column.

**Example 5.1** Let a linear transformation  $T: V_2 \rightarrow V_3$  be defined by  $T(x_1, x_2) = (x_1 + x_2, 2x_1 - x_2, 7x_2)$ .

If  $B_1 = \{e_1, e_2\}$  and  $B_2 = \{f_1, f_2, f_3\}$  are the standard bases of  $V_2$  and  $V_3$ , respectively, then

$$T(e_1) = f_1 + 2f_2 + 0f_3$$

and

$$T(e_2) = f_1 - f_2 + 7f_3.$$

The coordinate vectors of  $T(e_1)$  and  $T(e_2)$  are respectively

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}.$$

Therefore, the matrix of  $T$  relative to  $B_1$  and  $B_2$  is

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}.$$

**Example 5.2** Let a linear map  $T: V_3 \rightarrow V_3$  be defined by  $T(x_1, x_2, x_3) = (x_1 - x_2 + x_3, 2x_1 + 3x_2 - \frac{1}{2}x_3, x_1 + x_2 - 2x_3)$ .

If  $B_1 = \{e_1, e_2, e_3\}$ , the standard basis, and  $B_2 = \{(1, 1, 0), (1, 2, 3), (-1, 0, 1)\}$ , then  $T(e_1) = (1, 2, 1) = 2(1, 1, 0) + 0(1, 2, 3) + 1(-1, 0, 1)$ . So the coordinate vector of  $T(e_1)$  relative to  $B_2$  is  $(2, 0, 1)$ . Similarly, the coordinate vectors of  $T(e_2)$  and  $T(e_3)$  relative to  $B_2$  are  $(6, -3/2, 11/2)$  and  $(0, -1/4, -5/4)$ , respectively. Writing these successively as column vectors, we get the matrix of  $T$  relative to  $B_1$  and  $B_2$  as

$$\begin{bmatrix} 2 & 6 & 0 \\ 0 & -\frac{3}{2} & -\frac{1}{4} \\ 1 & \frac{11}{2} & -\frac{5}{4} \end{bmatrix}.$$

**Example 5.3** Let a linear map  $T: V_3 \rightarrow V_2$  be defined by  $T(e_1) = 2f_1 - f_2$ ,  $T(e_2) = f_1 + 2f_2$ ,  $T(e_3) = 0f_1 + 0f_2$ , where  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2\}$  are standard bases in  $V_3$  and  $V_2$ , respectively. Then the matrix of  $T$  relative to these bases is

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}.$$

Let us find the matrix of the same linear map  $T$  relative to some other bases, say  $B_1 = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$  and  $B_2 = \{(1, 1), (1, -1)\}$ . We have

$$\begin{aligned} T(x_1, x_2, x_3) &= x_1T(e_1) + x_2T(e_2) + x_3T(e_3) \\ &= (2x_1 + x_2)f_1 + (2x_2 - x_1)f_2 \\ &= (2x_1 + x_2, 2x_2 - x_1). \end{aligned}$$

$$\begin{aligned} \text{Therefore, } T(1, 1, 0) &= (3, 1) = 2(1, 1) + 1(1, -1) \\ T(1, 0, 1) &= (2, -1) = \frac{1}{2}(1, 1) + \frac{3}{2}(1, -1) \\ T(0, 1, 1) &= (1, 2) = \frac{3}{2}(1, 1) - \frac{1}{2}(1, -1). \end{aligned}$$

Hence, the matrix of  $T$  relative to  $B_1$  and  $B_2$  is

$$\begin{bmatrix} 2 & \frac{1}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Note that the matrix of  $T$  changes when we change the bases. We shall pursue this matter in Chapter 7.

**Example 5.4** Let a linear map  $T: \mathcal{P}_3 \rightarrow \mathcal{P}_2$  be defined by  $T(\alpha_0 + \alpha_1x + \alpha_2x^2 + \alpha_3x^3) = \alpha_3 + (\alpha_2 + \alpha_3)x + (\alpha_0 + \alpha_1)x^2$ . Let us calculate the

matrix of  $T$  relative to the bases  $B_1 = \{1, (x-1), (x-1)^2, (x-1)^3\}$  and  $B_2 = \{1, x, x^2\}$ :

$$\begin{aligned} T(1) &= x^2 = 0 \cdot 1 + 0x + 1x^2 \\ T((x-1)) &= 0 = 0 \cdot 1 + 0x + 0x^2 \\ T((x-1)^2) &= x - x^2 = 0 \cdot 1 + 1x - 1x^2 \\ T((x-1)^3) &= 1 - 2x + 2x^2 = 1 \cdot 1 - 2x + 2x^2. \end{aligned}$$

Hence, the matrix of  $T$  relative to  $B_1$  and  $B_2$  is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 2 \end{bmatrix}.$$

**Example 5.5** Let  $D: \mathcal{P}_3 \rightarrow \mathcal{P}_2$  be the differential map  $D(p) = p'$ . Let us calculate the matrix of  $D$  relative to the standard bases  $\{1, x, x^2, x^3\}$  and  $\{1, x, x^2\}$ :

$$\begin{aligned} D(1) &= 0 = 0 \cdot 1 + 0x + 0x^2 \\ D(x) &= 1 = 1 \cdot 1 + 0x + 0x^2 \\ D(x^2) &= 2x = 0 \cdot 1 + 2x + 0x^2 \\ D(x^3) &= 3x^2 = 0 \cdot 1 + 0x + 3x^2. \end{aligned}$$

Hence, the matrix of  $D$  relative to the standard bases is

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

**Example 5.6** We can easily check that the matrix of the identity map  $I: U \rightarrow U$  ( $\dim U = n$ ) relative to a basis  $B(B_1 = B_2 = B)$  is the  $n \times n$  matrix

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}. \quad (1)$$

This matrix is called the *identity matrix* and is denoted by  $I_n$  (or simply  $I$ , if  $n$  is understood).

The  $ij$ -th entry of this matrix is usually denoted by  $\delta_{ij}$ , where  $\delta_{ij}$ , called the *Kronecker delta*, is defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \quad (2)$$

So  $I_n$  is the matrix  $(\delta_{ij})_{n \times n}$ .

**Example 5.7** Using the same procedure as in Example 5.6, we can check

that, whatever the bases used, the zero map  $0: V_1 \rightarrow V_m$  will have the matrix

$$\begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \quad (3)$$

Each entry in this matrix is zero. This matrix is called the *zero matrix* of order  $m \times n$  and is denoted by  $0_{m \times n}$  or simply  $0$ , if the order is understood.

### Problem Set 5.1

In Problems 1 through 7 determine the matrix  $(T: B_1, B_2)$  for the given linear transformation  $T$  and the bases  $B_1$  and  $B_2$ .

- $T: V_2 \rightarrow V_2, T(x, y) = (x, -y)$

  - $B_1 = \{e_1, e_2\}, B_2 = \{(1, 1), (1, -1)\}$
  - $B_1 = \{(1, 1), (1, 0)\}, B_2 = \{(2, 3), (4, 5)\}$
- $T: V_3 \rightarrow V_2, T(x, y, z) = (x + y, y + z)$ .

  - $B_1 = \{(1, 1, 1), (1, 0, 0), (1, 1, 0)\}, B_2 = \{e_1, e_2\}$
  - $B_1 = \{(1, 1, \frac{2}{3}), (-1, 2, -1), (2, 3, \frac{1}{2})\}, B_2 = \{(\frac{1}{2}, 3), (\frac{1}{2}, 1)\}$ .
- $T: V_4 \rightarrow V_5, T(x_1, x_2, x_3, x_4) = (2x_1 + x_2, x_2 - x_3, x_3 + x_4, x_1, x_1 + x_2 + 3x_3 + x_4)$ .

$B_1 = \{(1, 2, 3, 1), (1, 0, 0, 1), (1, 1, 0, 0), (0, 1, 1, 1)\}$

$B_2 = \{e_1, e_2, e_3, e_4, e_5\}$
- $D: \mathcal{P}_n \rightarrow \mathcal{P}_n, D(p) = p'$

$B_1 = B_2 = \{1, x, x^2, \dots, x^n\}$ .
- $T: \mathcal{P}_4 \rightarrow \mathcal{P}_4, T(p)(x) = \int_1^x p'(t) dt$ .

  - $B_1 = B_2 = \{1, x, x^2, x^3, x^4\}$
  - $B_1 = \{1, x, x^2, x^3, x^4\}, B_2 = \{x - 1, x + 1, x^2 - x^4, x^3 + x^4, x^2 + x\}$ .
- $T: \mathcal{P}_2 \rightarrow \mathcal{P}_3, T(p) = xp(x)$ .

  - $B_1 = \{1, 1 + x, 1 - x + x^2\}, B_2 = \{1, 1 + x, x^2, 2x - x^3\}$
  - $B_1 = \{1, x, x^2\}, B_2 = \{1 + x, (1 + x)^2, (1 + x)^3, 1 - x\}$ .
- $T: V_2^{\mathbb{C}} \rightarrow V_2^{\mathbb{C}}, T(z_1, z_2) = (z_1 + iz_2, z_1 - iz_2)$ .  $B_1$  and  $B_2$  are standard bases.
- Determine, relative to the standard bases, the matrix of each linear transformation  $T$  in Problems 1(a) through 1(g) of Problem Set 4.2.
- True or false ?

- (a) Let  $T: V_4 \rightarrow V_6$  be a linear map. Then the matrix of  $T$  relative to any pair of bases is of order  $4 \times 6$ .
- (b) When referred to the standard basis, the matrix of the linear map  $T: V_2 \rightarrow V_2$  defined as  $(x, y) \mapsto (x', y')$ , where  $x = x' \cos \theta - y' \sin \theta$ ,  $y = x' \sin \theta + y' \cos \theta$ , is

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

- (c) The matrix of the identity map  $I_U: U \rightarrow U$  relative to any pair of bases is the identity matrix.

(d)  $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  is the matrix of the linear map  $T: V_3 \rightarrow V_3$

defined by  $T(e_1) = 0$ ,  $T(e_2) = e_1$ ,  $T(e_3) = 0$  relative to the standard bases.

(e)  $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$  is the matrix of the linear map  $T: V_3 \rightarrow V_3$  defined

by  $e_1 = T(e_3)$ ,  $e_2 = T(e_1)$ ,  $e_3 = T(e_2)$  relative to the standard bases.

- (f) To every linear transformation there corresponds a unique matrix.

## 5.2 LINEAR MAP ASSOCIATED WITH A MATRIX

In § 5.1 we started with a linear map and then defined its matrix. However, in order to define a matrix, it is not necessary to start from a linear map. Matrices can be considered as entities in their own rights as the following definition will show.

**5.2.1 Definition** Any rectangular array of numbers such as

$$\begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1j} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2j} & \cdots & \beta_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \beta_{i1} & \beta_{i2} & \cdots & \beta_{ij} & \cdots & \beta_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ \beta_{m1} & \beta_{m2} & \cdots & \beta_{mj} & \cdots & \beta_{mn} \end{bmatrix}.$$

is called an  $m \times n$  matrix. If  $m = n$ , the matrix is called a *square matrix*.

Two matrices  $A = (\alpha_{ij})_{m \times n}$  and  $B = (\beta_{ij})_{m \times n}$  are said to be *equal*, written as  $A = B$ , if  $\alpha_{ij} = \beta_{ij}$  for every  $i$  and  $j$ .

All the statements in Remark 5.1.2 also apply to a general matrix as defined in Definition 5.2.1.

A matrix whose entries are all real (complex) numbers is called a *real (complex) matrix*.

As already mentioned, our discussions in this chapter deal with real vector spaces, i.e.  $V_n$  and  $V_m$ . Also, the matrices used will be real. However, all the discussions hold for complex matrices if the real vector spaces  $V_n$  and  $V_m$  are replaced by the complex vector spaces  $V_n^C$  and  $V_m^C$ .

Let us start with a matrix  $B = (\beta_{ij})_{m \times n}$  and ask whether we can find a linear transformation  $T: U \rightarrow V$ , where  $U$  and  $V$  are suitable vector spaces with ordered bases, say  $B_1$  and  $B_2$ , so that  $B = (T: B_1, B_2)$ .

The answer to this question is *yes*, because the process of obtaining the matrix from a linear transformation (see Definition 5.1.1) is reversible. Let us go through the steps of this reverse process.

Let the matrix  $B = (\beta_{ij})_{m \times n}$  be given. Using this matrix  $B$ , we shall define a linear map  $S: U \rightarrow V$ , where  $U$  and  $V$  are vector spaces of dimensions  $n$  and  $m$ , respectively.

Let  $B_1 = \{u_1, u_2, \dots, u_n\}$  and  $B_2 = \{v_1, v_2, \dots, v_m\}$  be ordered bases for  $U$  and  $V$ , respectively. Then we define a linear map  $S: U \rightarrow V$  by prescribing values of  $S$  on the vectors of  $B_1$  as follows:

$$\begin{aligned} S(u_1) &= \beta_{11}v_1 + \beta_{21}v_2 + \dots + \beta_{m1}v_m, \\ &\vdots \\ S(u_j) &= \beta_{1j}v_1 + \beta_{2j}v_2 + \dots + \beta_{mj}v_m, \\ &\vdots \\ S(u_n) &= \beta_{1n}v_1 + \beta_{2n}v_2 + \dots + \beta_{mn}v_m. \end{aligned}$$

Extend  $S$  linearly to the whole of  $U$ . The linear transformation  $S$  is unique (Theorem 4.1.5). By this very construction, it is clear that  $B = (S: B_1, B_2)$ . For this purpose, we have only to note that the coordinate vector of  $S(u_j)$  relative to  $B_2$  is the  $j$ -th column of  $B$ .  $S: U \rightarrow V$  thus defined is called *the linear map associated with the matrix  $B$  relative to  $B_1$  and  $B_2$* .

Note that an  $m \times n$  matrix gives rise to a linear map from an  $n$ -dimensional vector space to an  $m$ -dimensional vector space. Further, note that, if we choose bases other than  $B_1$  and  $B_2$ , the linear map associated with the matrix  $B$  will also be different.

**Example 5.8** Consider a  $4 \times 3$  matrix

$$\begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -1 \\ -2 & 1 & 0 \\ 1 & 2 & -2 \end{bmatrix}.$$

This matrix will give rise to a linear map  $S: V_3 \rightarrow V_4$ . Suppose  $B_1 = \{u_1, u_2, u_3\}$  and  $B_2 = \{v_1, v_2, v_3, v_4\}$  are ordered bases for  $V_3$  and  $V_4$ , respectively. Write

$$\begin{aligned} S(u_1) &= 2v_1 + v_2 - 2v_3 + v_4 \\ S(u_2) &= -3v_1 + v_3 + 2v_4 \\ S(u_3) &= 4v_1 - v_2 - 2v_4. \end{aligned}$$

The linear extension of this definition of  $S$  on the basis elements is the required map  $S$ . If  $B_1$  and  $B_2$  are the standard bases  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2, f_3, f_4\}$  for  $V_3$  and  $V_4$ , respectively, then we get a linear transformation  $T: V_3 \rightarrow V_4$  given by

$$\begin{aligned} T(e_1) &= (2, 1, -2, 1), T(e_2) = (-3, 0, 1, 2), T(e_3) = (4, -1, 0, 2), \\ \text{i.e. } T(x_1, x_2, x_3) &= (2x_1 - 3x_2 + 4x_3, x_1 - x_3, -2x_1 + x_2, \\ &\quad x_1 + 2x_2 + 2x_3). \end{aligned}$$

Let  $\mathfrak{M}_{m,n}$  denote the set of all  $m \times n$  real matrices. Let  $U$  and  $V$  be real vector spaces of dimensions  $n$  and  $m$ , respectively. Fix ordered bases  $B_1$  for  $U$  and  $B_2$  for  $V$ . Then the process of determining the matrix of a linear map and the linear map corresponding to a matrix shows that the map

$$\tau: L(U, V) \rightarrow \mathfrak{M}_{m,n} \quad (1)$$

defined by  $\tau(T) = (T: B_1, B_2)$  is one-one and onto.

This result merely says that, if bases  $B_1$  and  $B_2$  are fixed, then to each linear map  $T: U \rightarrow V$  there exists a unique  $m \times n$  matrix and to each  $m \times n$  matrix there exists a unique linear map from  $U$  to  $V$ . This enables us to pass from linear transformation to matrix and vice versa.

When  $U = V_n$  and  $V = V_m$  and the bases  $B_1$  and  $B_2$  are the standard bases in the spaces, then the matrix associated with  $T: V_n \rightarrow V_m$  is called its *natural matrix*.

Note (cf Example 5.7) that  $\tau(0) = 0_{m \times n}$ . Further, note (cf Example 5.6) that, if

$$\tau: L(U) \rightarrow \mathfrak{M}_{n,n} \quad (2)$$

then  $\tau(I) = I_n$ , where  $I$  is the identity transformation on  $U$ , and  $I_n$  is the  $n \times n$  identity matrix.

### Problem Set 5.2 .

In Problems 1 through 4, for each given  $m \times n$  matrix  $A$  and bases  $B_1$  and  $B_2$ , determine a linear transformation  $T: V_n \rightarrow V_m$  such that  $A = (T: B_1, B_2)$ .

$$1. A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & -1 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

- (a)  $B_1$  and  $B_2$  are standard bases for  $V_4$  and  $V_3$ , respectively  
 (b)  $B_1 = \{(1, 1, 1, 2), (1, 1, 0, 0), (0, 0, 1, 1), (0, 1, 0, 0)\}$ ,  
 $B_2 = \{(1, 2, 3), (1, -1, 1), (2, 1, 1)\}$ .

$$2. A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

- (a)  $B_1 = B_2 = \{e_1, e_2, e_3\}$   
 (b)  $B_1 = \{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}$ ,  
 $B_2 = \{(1, 2, 3), (1, 1, 1), (2, 1, 1)\}$   
 (c)  $B_1 = \{(1, 2, 3), (1, -1, 1), (2, 1, 1)\}$ ,  
 $B_2 = \{(1, 1, 1), (1, 0, 0), (0, 1, 0)\}$ .

$$3. A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix}.$$

- (a)  $B_1$  and  $B_2$  are the standard bases for  $V_3$  and  $V_2$ , respectively  
 (b)  $B_1 = \{(1, 1, 1), (1, 2, 3), (1, 0, 0)\}$ ,  
 $B_2 = \{(1, 1), (1, -1)\}$   
 (c)  $B_1 = \{(1, 1, 1), (1, 2, 0), (0, -1, 0)\}$ ,  
 $B_2 = \{(1, 0), (2, 1)\}$ .

$$4. A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{bmatrix}.$$

- (a)  $B_1$  and  $B_2$  are standard bases for  $V_2$  and  $V_3$ , respectively  
 (b)  $B_1 = \{(1, 1), (-1, 1)\}$ ,  
 $B_2 = \{(1, 1, 1), (1, -1, 1), (0, 0, 1)\}$   
 (c)  $B_1 = \{(1, 2), (-2, 1)\}$ ,  
 $B_2 = \{(1, -1, -1), (1, 2, 3), (-1, 0, 2)\}$ .

5. If  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  is the matrix of a linear map  $T: V_2 \rightarrow V_2$  relative to the standard bases, then find the matrix of  $T^{-1}$  relative to the standard bases.

6. True or false ?

- (a) The matrix of the linear map  $T: U \rightarrow V$  is square iff  $\dim U = \dim V$ .  
 (b) An identity matrix is a square matrix.  
 (c) A zero matrix is a square matrix.



$$(d) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is the matrix of the identity transformation}$$

$I: V_3 \rightarrow V_3$  relative to the bases  $B_1 = \{e_1, e_2, e_3\}$  and  $B_2 = \{e_1, e_3, e_2\}$ .

$$(e) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \text{ determines a linear transformation from } V_3 \text{ to } V_3$$

defined as  $T(e_1) = f_1, T(e_2) = f_2, T(e_3) = f_1 + f_2$  relative to the standard bases  $\{e_1, e_2, e_3\}$  and  $\{f_1, f_2\}$ .

### 5.3 LINEAR OPERATIONS IN $M_{m,n}$

Throughout this article  $U$  and  $V$  are real vector spaces of dimensions  $n$  and  $m$ , respectively; and  $B_1 = \{u_1, u_2, \dots, u_n\}$  and  $B_2 = \{v_1, v_2, \dots, v_m\}$  are ordered bases for  $U$  and  $V$ , respectively.

Let  $T$  and  $S$  be two linear maps from  $U$  to  $V$ . Then  $T + S$  is also a linear map. If the matrices of  $T$  and  $S$  relative to  $B_1$  and  $B_2$  are  $A = (\alpha_{ij})_{m \times n}$  and  $B = (\beta_{ij})_{m \times n}$ , respectively, then

$$T(u_j) = \sum_{i=1}^m \alpha_{ij} v_i \quad (j = 1, 2, \dots, n) \quad (1)$$

and

$$S(u_j) = \sum_{i=1}^m \beta_{ij} v_i \quad (j = 1, 2, \dots, n). \quad (2)$$

So  $(T + S)(u_j) = T(u_j) + S(u_j)$

$$= \sum_{i=1}^m (\alpha_{ij} + \beta_{ij}) v_i \quad (j = 1, 2, \dots, n). \quad (3)$$

Therefore, the  $j$ -th column of the matrix of  $T + S$  relative to  $B_1$  and  $B_2$  is  $(\alpha_{1j} + \beta_{1j}, \alpha_{2j} + \beta_{2j}, \dots, \alpha_{ij} + \beta_{ij}, \dots, \alpha_{mj} + \beta_{mj})^T$ .

Hence, the matrix of  $T + S$  is

$$\begin{bmatrix} \alpha_{11} + \beta_{11} & \alpha_{12} + \beta_{12} & \dots & \alpha_{1j} + \beta_{1j} & \dots & \alpha_{1n} + \beta_{1n} \\ \alpha_{21} + \beta_{21} & \alpha_{22} + \beta_{22} & \dots & \alpha_{2j} + \beta_{2j} & \dots & \alpha_{2n} + \beta_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{i1} + \beta_{i1} & \alpha_{i2} + \beta_{i2} & \dots & \alpha_{ij} + \beta_{ij} & \dots & \alpha_{in} + \beta_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{m1} + \beta_{m1} & \alpha_{m2} + \beta_{m2} & \dots & \alpha_{mj} + \beta_{mj} & \dots & \alpha_{mn} + \beta_{mn} \end{bmatrix}. \quad (4)$$

This motivates the following definition of addition of matrices.

**5.3.1 Definition** Let  $A = (\alpha_{ij})$  and  $B = (\beta_{ij})$  be two  $m \times n$  matrices. Then the *sum*  $A + B$  is defined as the  $m \times n$  matrix  $(\alpha_{ij} + \beta_{ij})$ .

In other words, the sum of two matrices (of the same order) is obtained by adding the corresponding entries.

*Example 5.9* Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}.$$

Then 
$$A + B = \begin{bmatrix} 1 + (-1) & 3 + 2 & 2 + 1 \\ 0 + 3 & -1 + 1 & 1 + 2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 3 \\ 3 & 0 & 3 \end{bmatrix}.$$

**5.3.2 Remark** The argument preceding Definition 5.3.1 says only that the map  $\tau : L(U, V) \rightarrow M_{m,n}$  of § 5.2 preserves addition, i.e.  $\tau(T + S) = \tau(T) + \tau(S)$  for all  $T, S \in L(U, V)$ .

Let  $T : U \rightarrow V$  be a linear map and  $\alpha$  a scalar. Then  $\alpha T : U \rightarrow V$  is also a linear map. If the matrix of  $T$  relative to  $B_1$  and  $B_2$  is  $(\alpha_{ij})_{m \times n}$ , then

$$T(u_j) = \sum_{i=1}^m \alpha_{ij} v_i \quad (j = 1, 2, \dots, n). \quad (5)$$

So 
$$(\alpha T)(u_j) = \alpha T(u_j) = \sum_{i=1}^m (\alpha \alpha_{ij}) v_i \quad (j = 1, 2, \dots, n). \quad (6)$$

Therefore, the  $j$ -th column vector of the matrix of  $\alpha T$  relative to  $B_1$  and  $B_2$  is

$$(\alpha \alpha_{1j}, \alpha \alpha_{2j}, \dots, \alpha \alpha_{ij}, \dots, \alpha \alpha_{mj})^T.$$

Hence, the matrix of  $\alpha T$  is

$$\begin{bmatrix} \alpha \alpha_{11} & \alpha \alpha_{12} & \dots & \alpha \alpha_{1j} & \dots & \alpha \alpha_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha \alpha_{i1} & \alpha \alpha_{i2} & \dots & \alpha \alpha_{ij} & \dots & \alpha \alpha_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha \alpha_{m1} & \alpha \alpha_{m2} & \dots & \alpha \alpha_{mj} & \dots & \alpha \alpha_{mn} \end{bmatrix}. \quad (7)$$

This motivates the following definition of scalar multiplication for matrices.

**5.3.3 Definition** Let  $A = (\alpha_{ij})$  be an  $m \times n$  matrix and  $\alpha$  a scalar. Then the *scalar multiplication* of  $A$  by  $\alpha$  denoted by  $\alpha A$  is defined as the  $m \times n$  matrix  $(\alpha \alpha_{ij})$ .

In other words, multiplication of a matrix by a scalar  $\alpha$  is done by multiplying each entry by  $\alpha$ .

**Example 5.10** Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$(-\tfrac{1}{2})A = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} & -1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

**5.3.4 Remark** The argument preceding Definition 5.3.2 says only that the map  $\tau : L(U, V) \rightarrow \mathfrak{M}_{m, n}$  of § 5.2 preserves scalar multiplication, i.e.  $\tau(\alpha T) = \alpha \tau(T)$  for all  $T \in L(U, V)$ .

From the definition of addition and scalar multiplication it immediately follows that  $\mathfrak{M}_{m, n}$  is a real vector space. (The reader is advised to go through the various steps of verification.) Note that the zero element of this vector space is the zero matrix  $0_{m \times n}$  of Example 5.7.

Further, Remarks 5.3.2 and 5.3.4 show that the map  $\tau : L(U, V) \rightarrow \mathfrak{M}_{m, n}$  is linear. Since we have already proved in § 5.2 that  $\tau$  is one-one and onto, it follows that  $\tau$  is an isomorphism. We collect all these results in the form of a theorem.

**5.3.5 Theorem** (a)  $\mathfrak{M}_{m, n}$  is a real vector space for the foregoing definitions of addition and scalar multiplication.

(b)  $\tau : L(U, V) \rightarrow \mathfrak{M}_{m, n}$  defined by  $\tau(T) = (T : B_1, B_2)$  is an isomorphism.

The implication of this theorem is that, for all practical purposes of linear algebra, linear transformations between finite-dimensional vector spaces are just matrices and vice versa.

We shall now determine the dimension of  $L(U, V)$ . By Theorem 5.3.5, it is the same as the dimension of  $\mathfrak{M}_{m, n}$ , because isomorphic vector spaces have the same dimension. (Why?) So we have only to prove the following theorem.

**5.3.6 Theorem** The dimension of the vector space  $\mathfrak{M}_{m, n}$  is  $mn$ .

*Proof:* Given  $i$  and  $j$ , define the matrix  $E_{ij}$  as the  $m \times n$  matrix with 1 in the  $ij$ -th entry and zero in all the other entries. We claim that the set  $B$  of matrices

$$\begin{aligned} B &= \{E_{11}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{m1}, \dots, E_{mn}\} \\ &= \{E_{ij} \in \mathfrak{M}_{m, n} \mid i = 1, \dots, m; j = 1, \dots, n\} \end{aligned}$$

is a basis for  $\mathfrak{M}_{m, n}$ .

To prove that  $B$  is li in  $\mathfrak{M}_{m, n}$ , we assume  $\alpha_{11}E_{11} + \dots + \alpha_{1n}E_{1n} + \alpha_{21}E_{21} + \dots + \alpha_{2n}E_{2n} + \dots + \alpha_{m1}E_{m1} + \dots + \alpha_{mn}E_{mn} = 0_{m \times n}$ . This means that

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

Hence,  $\alpha_{ij} = 0$  for all  $i = 1, \dots, m$  and  $j = 1, \dots, n$ . Thus,  $B$  is LI. Again, to prove that  $[B] = \mathcal{M}_{m,n}$ , let us take a general matrix  $A = (\alpha_{ij})$  in  $\mathcal{M}_{m,n}$ . Then it can be easily verified that

$$A = \alpha_{11}E_{11} + \dots + \alpha_{1n}E_{1n} + \alpha_{21}E_{21} + \dots + \alpha_{2n}E_{2n} + \alpha_{m1}E_{m1} + \dots + \alpha_{mn}E_{mn}.$$

Hence,  $B$  is a basis. Hence the theorem. ■

The basis  $\{E_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$  is called the *standard basis* for  $\mathcal{M}_{m,n}$ .

Combining Theorems 5.3.5 and 5.3.6, we have the following corollary.

**5.3.7 Corollary** *If  $U$  and  $V$  are finite-dimensional vector spaces, then*  
 $\dim L(U, V) = \dim U \times \dim V$ .

### Problem Set 5.3

In Problems 1 through 4 determine  $\alpha A + \beta B$  for the given scalars  $\alpha$  and  $\beta$  and the matrices  $A$  and  $B$ .

1.  $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix}.$

(a)  $\alpha = 2, \beta = 7$       (b)  $\alpha = 3, \beta = -2$ .

2.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}.$

(a)  $\alpha = 3, \beta = 5$       (b)  $\alpha = 2, \beta = -3$ .

3.  $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 1 & 1 \\ 3 & 1 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 3 & -1 & 2 & 0 \\ 1 & 5 & 7 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix}.$

(a)  $\alpha = 2, \beta = -6$       (b)  $\alpha = 3, \beta = 5$

(c)  $\alpha = -7, \beta = 3$ .

4.  $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & -1 \\ 3 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 1 & -1 \end{bmatrix}.$

- (a)  $\alpha = 7, \beta = -5$       (b)  $\alpha = 1/2, \beta = 2/3$   
 (c)  $\alpha = 1/3, \beta = 4/5$ .

5. Let  $T_1, T_2 : U \rightarrow V$  be two linear maps. Let  $B_1, B_2$  be ordered bases for  $U$  and  $V$ , respectively. Then prove that

$$(\alpha_1 T_1 + \alpha_2 T_2 : B_1, B_2) = \alpha_1 (T_1 : B_1, B_2) + \alpha_2 (T_2 : B_1, B_2).$$

6. Let  $S, T : V_3 \rightarrow V_4$  be defined as

$$S(x_1, x_2, x_3) = (x_1 + x_2, x_1 - 2x_2 + x_3, x_2 + 3x_3, x_1 + x_3)$$

$$T(x_1, x_2, x_3) = (x_1 + 2x_2, x_1 - x_2, 3x_2 + x_3, x_1 + x_2 + x_3).$$

Determine the matrix of  $3S - 4T$  relative to the standard bases by two different methods.

7. Let  $S, T : \mathcal{P}_3 \rightarrow \mathcal{P}_4$  be defined as

$$(S(p))(x) = (x^2 - 1)p'(x)$$

$$(T(p))(x) = (3x + 2)p(x) - \int_1^x p'(t)dt.$$

Determine the matrix of  $4S + 2T$  relative to the ordered bases  $B_1 = \{1, x, x^2, x^3\}$  and  $B_2 = \{(1-x), (1+x), (1-x)^2, (1-x)^3, x^4/2\}$  by two different methods.

8. Let  $B_1 = \{u_1, u_2, \dots, u_n\}$  and  $B_2 = \{v_1, v_2, \dots, v_m\}$  be ordered bases for the vector spaces  $U$  and  $V$ , respectively. Define  $T_{ij} : U \rightarrow V$ ,  $1 \leq i \leq m, 1 \leq j \leq n$  such that

$$T_{ij}(u_k) = \begin{cases} 0 & \text{if } k \neq i \\ v_j & \text{if } k = i. \end{cases}$$

Then prove that

(a)  $(T_{ij} : B_1, B_2) = E_{ij}$  (cf Theorem 5.3.6)

(b)  $\{T_{ij}\}$  is a basis for  $L(U, V)$ .

9. Define  $T : \mathcal{M}_{2,2} \rightarrow \mathcal{M}_{2,2}$  such that

$$T \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11} + \alpha_{12} & 0 & \alpha_{12} + \alpha_{22} \\ \alpha_{12} & \alpha_{21} + \alpha_{22} & 0 \end{bmatrix}.$$

Prove that  $T$  is linear and determine its matrix relative to the standard bases for  $\mathcal{M}_{2,2}$  and  $\mathcal{M}_{2,2}$ .

10. Repeat Problem 9 for  $T : \mathcal{M}_{2,2} \rightarrow \mathcal{M}_{2,2}$  defined as

$$T \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{bmatrix} = \begin{bmatrix} \alpha_{11} + \alpha_{12} + \alpha_{13} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} + \alpha_{23} \end{bmatrix}.$$

11. Let  $V$  be the subspace of  $\mathcal{C}^{(m)}(-\infty, \infty)$  spanned by the functions  $\sin x, \cos x, \sin x \cos x, \sin^2 x, \cos^2 x$ . Determine the dimension of  $V$ , and prove that the differential operator  $D^n$  maps  $V$  into itself for every positive integer  $n \leq m$ .

Determine the matrix of (a)  $2D + 3$ , (b)  $3D^2 - D + 4$  relative to the basis of  $V$  obtained from the given spanning set of  $V$ .

## 12. True or false ?

$$(a) \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 6 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 9 & 1 \\ 9 & 4 & 7 \end{bmatrix}.$$

$$(b) \begin{bmatrix} \alpha & 3\alpha \\ 2\beta & 4\beta \end{bmatrix} = \alpha\beta \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

(c)  $0A = 0_{m \times n}$  for any  $m \times n$  matrix  $A$

(d) The set of all square matrices is a vector space.

(e) Every matrix can be written as the sum of two matrices, one of which is two times the other.

(f) The set of all  $3 \times 3$  matrices of the form

$$\begin{bmatrix} 0 & \alpha & \beta \\ \gamma & 0 & \delta \\ \theta & \rho & 0 \end{bmatrix}$$

is a group under the operation of addition.

## 5.4 MATRIX MULTIPLICATION

Hereafter, we shall use only standard bases in the vector spaces  $V_n$ , unless otherwise stated.

In this article let  $S: V_p \rightarrow V_n$  and  $T: V_n \rightarrow V_m$  be linear maps. Let  $B_1 = \{e_1, e_2, \dots, e_n\}$ ,  $B_2 = \{f_1, f_2, \dots, f_m\}$ , and  $B_3 = \{g_1, g_2, \dots, g_p\}$  be the standard bases for  $V_n$ ,  $V_m$ , and  $V_p$ , respectively.

Let the natural matrix for  $T$  be  $A = (\alpha_{ij})_{m \times n}$ , and that of  $S$  be  $B = (\beta_{ij})_{n \times p}$ .

We know that  $TS: V_p \rightarrow V_m$  is also a linear map. Let the natural matrix of  $TS$  be  $C = (\gamma_{ij})_{m \times p}$ . We shall now calculate  $\gamma_{ij}$  in terms of  $\alpha$ 's and  $\beta$ 's. We have, for each  $k = 1, 2, \dots, n$ ,

$$\begin{aligned} T(e_k) &= \sum_{i=1}^m \alpha_{ik} f_i \\ &= \alpha_{1k} f_1 + \alpha_{2k} f_2 + \dots + \alpha_{ik} f_i + \dots + \alpha_{mk} f_m \end{aligned} \quad (1)$$

and, for each  $j = 1, 2, \dots, p$ ,

$$\begin{aligned} S(g_j) &= \sum_{k=1}^n \beta_{kj} e_k \\ &= \beta_{1j} e_1 + \beta_{2j} e_2 + \dots + \beta_{kj} e_k + \dots + \beta_{nj} e_n. \end{aligned} \quad (2)$$

Also, for each  $j = 1, 2, \dots, p$ ,

$$\begin{aligned} (TS)(g_j) &= \sum_{i=1}^m \gamma_{ij} f_i \\ &= \gamma_{1j} f_1 + \gamma_{2j} f_2 + \dots + \gamma_{ij} f_i + \dots + \gamma_{mj} f_m. \end{aligned} \quad (3)$$

Now, for each  $j = 1, 2, \dots, p$ , we also have

$$\begin{aligned}(TS)(g_j) &= T(S(g_j)) \\ &= T(\beta_{1j}e_1 + \beta_{2j}e_2 + \dots + \beta_{kj}e_k + \dots + \beta_{nj}e_n) \quad (\text{by (2)}) \\ &= \beta_{1j}T(e_1) + \beta_{2j}T(e_2) + \dots + \beta_{kj}T(e_k) \\ &\quad + \dots + \beta_{nj}T(e_n), \quad (4)\end{aligned}$$

because  $T$  is linear. Each of the terms on the right-hand side of Equation (4) is, by Equation (1), a linear combination of the vectors  $f_1, f_2, \dots, f_m$ . Collecting the terms involving  $f_i$ , we see that the coefficient of  $f_i$  on the right-hand side of Equation (4) is

$$\begin{aligned}&\beta_{1j}\alpha_{i1} + \beta_{2j}\alpha_{i2} + \dots + \beta_{kj}\alpha_{ik} + \dots + \beta_{nj}\alpha_{in} \\ &= \alpha_{i1}\beta_{1j} + \alpha_{i2}\beta_{2j} + \dots + \alpha_{ik}\beta_{kj} + \dots + \alpha_{in}\beta_{nj} \\ &= \sum_{k=1}^n \alpha_{ik}\beta_{kj}.\end{aligned} \quad (5)$$

Comparing this with the coefficient of  $f_i$  on the right-hand side of Equation (3), we get

$$\gamma_{ij} = \sum_{k=1}^n \alpha_{ik}\beta_{kj} \quad (6)$$

for every  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, p$ .

Using this as motivation, we define the product of two matrices as follows.

**5.4.1 Definition** Let  $A = (\alpha_{ij})$  be an  $m \times n$  matrix and  $B = (\beta_{ij})$  an  $n \times p$  matrix. Then the *product*  $AB$  is defined as the  $m \times p$  matrix  $C = (\gamma_{ij})$ , where

$$\gamma_{ij} = \sum_{k=1}^n \alpha_{ik}\beta_{kj}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, p. \quad (7)$$

In order to understand and use this definition, we shall define the 'inner product' of two vectors in  $V_n$ .

**5.4.2 Definition** Let  $u = (x_1, x_2, \dots, x_n)$  and  $v = (y_1, y_2, \dots, y_n)$  be two vectors in  $V_n$ . Then the *inner product* of  $u$  and  $v$  denoted by  $u \cdot v$  is the scalar

$$u \cdot v = x_1y_1 + x_2y_2 + \dots + x_ny_n.$$

For example, if  $u = (1, 2, -3)$  and  $v = (0, -1, 2)$ , then  $u \cdot v = 1 \times 0 + 2 \times (-1) + (-3) \times 2 = -8$ .

By this definition, the expression for  $\gamma_{ij}$  in Equation (7) becomes

$$\begin{aligned}\gamma_{ij} &= \alpha_{i1}\beta_{1j} + \alpha_{i2}\beta_{2j} + \dots + \alpha_{in}\beta_{nj} \\ &= (\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}) \cdot (\beta_{1j}, \beta_{2j}, \dots, \beta_{nj}) \\ &= (i\text{-th row vector of } A) \cdot (j\text{-th column vector of } B), \quad (8)\end{aligned}$$

both being considered vectors in  $V_n$ .

So Definition 5.4.1 can be reworded as follows,

**5.4.3 Definition (Reworded)** Let  $A = (\alpha_{ij})$  be an  $m \times n$  matrix and  $B = (\beta_{ij})$  an  $n \times p$  matrix. Then the *product*  $AB$  is the  $m \times p$  matrix  $C = (\gamma_{ij})$ , where  $\gamma_{ij}$  is the inner product of the  $i$ -th row vector of  $A$  and the  $j$ -th column vector of  $B$ , for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, p$ , both being treated like vectors in  $V_n$ .

**Example 5.11** Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 1 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ -2 & 3 \end{bmatrix}.$$

Here  $A$  is a  $4 \times 3$  matrix and  $B$  is a  $3 \times 2$  matrix. So  $AB$  is the  $4 \times 2$  matrix

$$\begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \\ \gamma_{31} & \gamma_{32} \\ \gamma_{41} & \gamma_{42} \end{bmatrix},$$

where

$$\begin{aligned} \gamma_{11} &= (\text{1st row of } A) \cdot (\text{1st column of } B) \\ &= 3 \times 1 + 1 \times 0 + 2 \times (-2) = -1 \end{aligned}$$

$$\begin{aligned} \gamma_{12} &= (\text{1st row of } A) \cdot (\text{2nd column of } B) \\ &= 3 \times (-1) + 1 \times 1 + 2 \times 3 = 4 \end{aligned}$$

$$\begin{aligned} \gamma_{21} &= (\text{2nd row of } A) \cdot (\text{1st column of } B) \\ &= 0 \times 1 + (-1) \times 0 + 1 \times (-2) = -2, \end{aligned}$$

and so on. Completing the calculations for all the  $\gamma$ 's, we find that

$$AB = \begin{bmatrix} -1 & 4 \\ -2 & 2 \\ 5 & -7 \\ 2 & -4 \end{bmatrix}.$$

**Example 5.12** Let

$$A = \begin{bmatrix} 1+i & -i & 2 \\ 4+i & 1-i & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1-i \\ 1 & 2 \\ 1-i & i \end{bmatrix}$$

Here  $A$  is a  $2 \times 3$  matrix and  $B$  is a  $3 \times 2$  matrix. So  $AB$  is the  $2 \times 2$  matrix



$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix},$$

where

$$\begin{aligned} \alpha_{11} &= (1+i)0 - i(1) + 2(1-i) = 2 - 3i \\ \alpha_{12} &= (1+i)(1-i) - i(2) + 2i = 2 \\ \alpha_{21} &= (4+i)0 + (1-i)1 + 0(1-i) = 1 - i \\ \alpha_{22} &= (4+i)(1-i) + (1-i)2 + 0i = 7 - 5i. \end{aligned}$$

So

$$AB = \begin{bmatrix} 2 - 3i & 2 \\ 1 - i & 7 - 5i \end{bmatrix}.$$

**5.4.4 Remark** The discussion preceding Definition 5.4.1 proves that if  $\tau: L(V_n) \rightarrow \mathfrak{M}_{n,n}$  is defined by  $\tau(T) =$  the natural matrix of  $T$ , then  $\tau(TS) = \tau(T)\tau(S)$  for all  $T, S \in L(V_n)$  and  $\tau^{-1}(AB) = \tau^{-1}(A)\tau^{-1}(B)$  for all  $A, B \in \mathfrak{M}_{n,n}$ .

In other words, the composition of transformations corresponds to matrix multiplication.

Let us further analyse the process of matrix multiplication, namely,

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots \\ \alpha_{i1} & \alpha_{i2} & \dots & \alpha_{ik} & \dots & \alpha_{in} \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \dots \beta_{1j} \dots \\ \dots \beta_{2j} \dots \\ \vdots \\ \dots \beta_{kj} \dots \\ \vdots \\ \dots \beta_{nj} \dots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ \dots \gamma_{ij} \dots \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

Here attention is focussed on the fact that the  $ij$ -th element of  $AB$  is ( $i$ -th row of  $A$ )  $\cdot$  ( $j$ -th column of  $B$ ). But recall that we can form the inner product of two vectors only if both have the same number of coordinates. In other words, in order that matrix multiplication be possible, we should have the number of entries in the  $i$ -th row of  $A$  equal to the number of entries in the  $j$ -th column of  $B$ , i.e. the number of columns of  $A$  should be the same as the number of rows of  $B$ . Thus, an  $m \times n$  matrix  $A$  can be multiplied on the right by a  $q \times p$  matrix  $B$  iff  $n = q$ . If  $n \neq q$ , then the product  $AB$  is not defined.

A  $1 \times n$  matrix  $A$  is simply a row vector of the form  $(\alpha_1, \alpha_2, \dots, \alpha_n)$ . An  $n \times 1$  matrix  $B$  is simply a column vector of the form  $(\beta_1, \beta_2, \dots, \beta_n)^T$ . The product  $AB$  of these two matrices is a  $1 \times 1$  matrix  $[\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_n\beta_n]$ , whose only entry is nothing but a scalar, which is the inner product of vectors  $A$  and  $B$ .

In general, matrix multiplication is not commutative. For, if

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, D = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 3 & 0 \\ 7 & -1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}.$$

Hence,  $AB \neq BA$ . But we find that  $CD = \begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix} = DC$ .

On the other hand, we have the following theorem.

**5.4.5 Theorem** (a) *Matrix multiplication is associative, that is,  $(AB)C = A(BC)$ , whenever either side is defined.*

(b) *Matrix multiplication is distributive over addition, that is,  $A(B + C) = AB + AC$ , whenever either side is defined. Also,  $(B + C)D = BD + CD$ , whenever either side is defined.*

The proof is left to the reader. Note the following:

(i)  $0_{m \times n} A_{n \times p} = 0_{m \times p}$  and

$$A_{n \times p} 0_{p \times m} = 0_{n \times m}.$$

(ii)  $I_n A_{n \times m} = A_{n \times m}$  and

$$A_{n \times m} I_m = A_{n \times m}$$

**5.4.6 Definition** If  $A$  and  $B$  are square matrices of the same order such that  $AB = I = BA$ , then  $B$  is called an *inverse of  $A$* .

**Example 5.13** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = BA.$$

Therefore,  $B$  is an inverse of  $A$  and  $A$  is an inverse of  $B$ .

**5.4.7 Theorem** *If an inverse of a matrix  $A$  exists, then it is unique.*

**Proof:** Suppose that  $B$  and  $C$  are two inverses of  $A$ . Then  $AB = I = BA$  and  $AC = I = CA$ . Thus, we have

$$\begin{aligned} C &= CI = C(AB) \\ &= (CA)B && \text{(Theorem 5.4.5(a))} \\ &= IB = B. \end{aligned}$$

Hence,  $B = C$ . ■

In view of this theorem, we speak of *the* (unique) *inverse* of  $A$  and denote it by  $A^{-1}$ . Whenever  $A^{-1}$  exists,  $A$  is said to be *invertible*.

By Theorem 5.4.7, it follows that if  $A^{-1}$  is the inverse of  $A$ , then  $A$  is the inverse of  $A^{-1}$ , i.e.  $(A^{-1})^{-1} = A$ . Both the matrices are therefore invertible.

**5.4.8 Theorem** *An  $n \times n$  matrix  $A$  is invertible iff the corresponding linear transformation  $T$  (via the standard bases) is nonsingular.*

*Proof:* Suppose  $A$  is invertible. Then there exists an  $n \times n$  matrix  $B$  such that  $AB = I_n = BA$ . Let the linear transformation corresponding to  $B$  be  $S: V_n \rightarrow V_n$ . Then, by Remark 5.4.4,

$$\tau^{-1}(AB) = \tau^{-1}(I_n) = \tau^{-1}(BA)$$

$$\text{or} \quad \tau^{-1}(A)\tau^{-1}(B) = \tau^{-1}(I_n) = \tau^{-1}(B)\tau^{-1}(A)$$

$$\text{or} \quad TS = I = ST.$$

So, by Theorem 4.7.2,  $T$  is nonsingular.

Conversely, if  $T$  is nonsingular, then there exists a linear transformation  $S: V_n \rightarrow V_n$  such that  $TS = I = ST$ . Therefore, by Remark 5.4.4,  $\tau(TS) = \tau(I) = \tau(ST)$ . If  $B$  is the natural matrix of  $S$ , we have  $AB = I_n = BA$ . Hence,  $A$  is invertible. ■

For this reason, a square matrix that is invertible is called a *nonsingular matrix*. A square matrix that is not invertible is called a *singular matrix*.

The foregoing ideas can be used in solving certain linear systems of equations. To do this, we first write the system in matrix form.

Suppose that

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \dots, x_n)^T$$

Then  $Ax$  is the  $m \times 1$  matrix, i.e. a column vector with  $m$  entries:

$$Ax = \begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1n}x_n \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \dots + \alpha_{2n}x_n \\ \vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \dots + \alpha_{mn}x_n \end{bmatrix}.$$

Further, suppose we have  $m$  linear equations in  $n$  unknowns:



$$(e) \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \begin{bmatrix} 2 \\ i \end{bmatrix}.$$

3. For the given matrices  $A$  and  $B$ , determine the product  $AB$  or  $BA$ , whichever exists.

$$(a) A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 3 & 2 \\ 1 & 5 \end{bmatrix}.$$

$$(b) A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -5 & 7 \\ 3 & 4 \end{bmatrix}$$

$$(c) A = \begin{bmatrix} 1 & 5 & 7 \\ 3 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 7 & 0 \end{bmatrix}$$

$$(d) A = \begin{bmatrix} 3 & 5 \\ 7 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 3 & 0 & 1 \end{bmatrix}$$

$$(e) A = \begin{bmatrix} 1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 2 & 5 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 5 & 7 \\ -1 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$(f) A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix}.$$

4. Determine  $A^{-1}$  for the given matrix  $A$  in each of the following :

$$(a) A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (b) A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad (c) A = \begin{bmatrix} \alpha & \beta \\ \beta & 0 \end{bmatrix}, \beta \neq 0$$

$$(d) A = \begin{bmatrix} \alpha & 0 \\ \beta & \gamma \end{bmatrix}, \alpha\gamma \neq 0 \quad (e) A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

5. Compute  $A^2$ ,  $A^3$ , and  $A^4$  for each given square matrix  $A$ .

$$(a) \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 3 & -1 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 5 & 7 \end{bmatrix}.$$

6. Determine all  $2 \times 2$  matrices that commute with

$$(a) \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$(d) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

7. If  $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ ,

$$D = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}, \text{ solve the following matrix}$$

equations :

$$(a) 3X - AB = C \quad (b) (BXD + AXD)E = C^2$$

$$(c) EXD = I.$$

$$8. A = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, B = \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \mu_n \end{bmatrix}.$$

Compute (a)  $AB$  (b)  $BA$  (c)  $A^{-1}$ , if it exists.

9. An  $n \times n$  matrix  $A$  is said to be *nilpotent* if  $A^n = 0$  for some positive integer  $n$ . The smallest positive integer  $n$ , for which  $A^n = 0$ , is called the *degree of nilpotence* of  $A$ .

Check whether the following matrices are nilpotent. In the case of nilpotent matrices find the degree of nilpotence.

$$(a) \begin{bmatrix} 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ \beta_1 & \beta_2 & 0 & 0 \\ \gamma_1 & \gamma_2 & \gamma_3 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 5 & -2 \\ 1 & 2 & -1 \\ 3 & 6 & -3 \end{bmatrix} \quad (d) \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & -4 \\ 1 & -3 & -4 \end{bmatrix}.$$

10. If  $A$  and  $B$  are square matrices of the same order, then prove that

(a)  $A^2 - B^2 = (A - B)(A + B)$  iff  $AB = BA$ .

(b)  $A^2 \pm 2AB + B^2 = (A \pm B)^2$  iff  $AB = BA$ .

11. Denote the matrix  $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$

as  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ , where  $A_{11}$  is the matrix  $[\alpha_{11}]$ ,  $A_{12} = [\alpha_{12} \ \alpha_{13}]$ ,

$A_{21} = \begin{bmatrix} \alpha_{21} \\ \alpha_{31} \end{bmatrix}$ ,  $A_{22} = \begin{bmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{bmatrix}$ . Similarly, define  $B, B_{11}, \dots$  by

replacing  $\alpha_i$  by  $\beta_i$ . Then prove that

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

and  $AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$ .

12. If  $\alpha, \beta$  are any scalars, then prove that  $A^2 - (\alpha + \beta)A + \alpha\beta I = (A - \alpha I)(A - \beta I)$ , where  $A$  is any square matrix of order  $n$  and  $I = I_n$ .

13. If  $\alpha, \beta$  are scalars such that  $A = \alpha B + \beta I$ , then prove that  $AB = BA$ .

14. A square matrix  $A$  is said to be *involutory* if  $A^2 = I$ . Prove that the

matrices  $\begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}$  are involutory for all scalars  $\alpha$ .

Determine all  $2 \times 2$  involutory matrices.

15. Let  $\tau : L(V_n) \rightarrow \mathfrak{M}_{n,n}$  be defined as in expression (1), § 5.2. Then prove that  $\tau^{-1}(AB) = \tau^{-1}(A)\tau^{-1}(B)$  for all  $A, B \in \mathfrak{M}_{n,n}$ .

16. Let  $p(x) = \alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$  be a polynomial of degree  $n$ , and  $A$  be a square matrix of order  $m$ . Then the *matrix polynomial*  $p(A)$  is defined as  $p(A) = \alpha_0 I + \alpha_1 A + \dots + \alpha_n A^n$ , where  $I = I_m$ .

If  $f(x) = 7x^2 - 3x + 5$ ,  $g(x) = 3x^3 - 2x^2 + 5x - 1$ ,

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \text{ and } B = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \text{ evaluate}$$

- (a)  $f(A)$  (b)  $g(A)$  (c)  $f(B)$  (d)  $g(B)$  (e)  $f(2A + 3B)$   
 (f)  $g(3A - 7B)$ .

17. Prove Theorem 5.4.5.

18. Using matrix methods, solve the following systems of linear equations :

$$\begin{array}{lll} \text{(a)} & x + 2y = 3 & \text{(b)} \quad \alpha x + \beta y = a \\ & y = 1 & \beta x = b \end{array} \quad \begin{array}{l} \text{(c)} \quad x - 2z = 1 \\ 2x + y = 2 \\ x + 2z = 3. \end{array}$$

19. If  $A$  and  $B$  are two nonsingular matrices of the same order, then prove that  $AB$  is nonsingular and  $(AB)^{-1} = B^{-1}A^{-1}$ .

20. Prove that the set of the following eight matrices forms a group for matrix multiplication :

$$\begin{aligned} I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, D_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\ D_2 &= \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, D_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \\ D_4 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, D_5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \\ D_6 &= \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, D_7 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \end{aligned}$$

Write the multiplication table of the group.

21. True or false ?

$$\text{(a)} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a^2 + b^2 & ac + bd \\ -ac + bd & c^2 + d^2 \end{bmatrix}.$$

- (b) If  $AB = I$ , then  $A$  is invertible.  
 (c) If  $A$  and  $B$  are square matrices of the same order, then  $AB = BA$ .  
 (d) The system of two equations  $2x + 3y = 1$  and  $4x + 6y = 2$  cannot be solved by the method discussed in this article.  
 (e) Let  $A$  be  $2 \times 3$  and  $B$  be  $3 \times 2$  matrices. Further, let  $C$  be  $2 \times 2$  and  $D$  be  $3 \times 3$  matrices. Then  
     (i)  $AB$  is defined but not  $BA$ .  
     (ii)  $AB$  and  $BA$  are both defined.



- (iii)  $BC$  is defined but not  $CB$ .  
 (iv)  $DA$  is defined but not  $AD$ .  
 (f) For every integer  $n > 1$ , there exists a nonzero  $n \times n$  matrix which is nilpotent.  
 (g) For every integer  $n > 0$ , there exists a nonzero  $n \times n$  matrix which is nilpotent.

## 5.5 RANK AND NULLITY OF A MATRIX

We have seen that every  $m \times n$  matrix  $A$  corresponds to a unique linear transformation  $T: V_n \rightarrow V_m$ , where we have agreed to use the standard bases in both spaces. We shall now show that the value of  $T$  at an arbitrary vector  $x$  is just the effect of matrix multiplication of  $A$  by  $x$  in that order.

Let  $A = (\alpha_{ij})_{m \times n}$ . Let the standard bases in  $V_n$  and  $V_m$  be  $\{e_1, e_2, \dots, e_n\}$  and  $\{f_1, f_2, \dots, f_m\}$ , respectively. If  $x = (x_1, x_2, \dots, x_n)$  is an arbitrary element of  $V_n$ , then  $x = x_1e_1 + x_2e_2 + \dots + x_ne_n$ . Let  $T: V_n \rightarrow V_m$  be the linear transformation associated with  $A$ . Then, for each  $j = 1, 2, \dots, n$ ,

$$T(e_j) = \alpha_{1j}f_1 + \alpha_{2j}f_2 + \dots + \alpha_{mj}f_m = (\alpha_{1j}, \alpha_{2j}, \dots, \alpha_{mj}), \quad (1)$$

because  $\{f_1, f_2, \dots, f_m\}$  is the standard basis for  $V_m$ . Writing these  $T(e_j)$ 's as column vectors, we get

$$T(x) = T(x_1e_1 + x_2e_2 + \dots + x_ne_n) = x_1T(e_1) + x_2T(e_2) + \dots + x_nT(e_n)$$

$$= x_1 \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{m1} \end{bmatrix} + x_2 \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \vdots \\ \alpha_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{mn} \end{bmatrix} \quad (2)$$

$$= \begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1n}x_n \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \dots + \alpha_{2n}x_n \\ \vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \dots + \alpha_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = Ax. \quad (3)$$

Thus, the vector  $T(x)$  is obtained by multiplying the matrix  $A$  by the

column vector  $x = (x_1, x_2, \dots, x_n)^T$  in that order. In this sense therefore a matrix can be identified with the linear transformation it represents. In other words, if  $A$  is an  $m \times n$  matrix, then the corresponding linear transformation  $T: V_n \rightarrow V_m$  represents the operation of matrix multiplication of  $A$  by vectors of the space  $V_n$  (assuming that we use standard bases throughout).

Let  $A$  be an  $m \times n$  matrix  $(\alpha_{ij})$ . Instead of talking of the linear transformation  $T$  associated with  $A$ , we talk of  $A$  itself as the linear transformation, the understanding being that the value of the linear transformation  $A$  at  $x = (x_1, x_2, \dots, x_n)^T$  is just the matrix product  $Ax$ .

**5.5.1 Definition** The range, kernel, rank, and nullity of a matrix  $A$  are defined as the range, kernel, rank, and nullity of the linear transformation  $A$ .

**Range of  $A$**  The range of  $A: V_n \rightarrow V_m$  is the set of all vectors of the form  $Ax \in V_m$ ,  $x \in V_n$ . For  $A = (\alpha_{ij})$ , it is the set of all column vectors of the form

$$\begin{bmatrix} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1n}x_n \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \dots + \alpha_{2n}x_n \\ \vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \dots + \alpha_{mn}x_n \end{bmatrix},$$

i.e.

$$x_1 \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \vdots \\ \alpha_{m1} \end{bmatrix} + x_2 \begin{bmatrix} \alpha_{12} \\ \alpha_{22} \\ \vdots \\ \alpha_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} \alpha_{1n} \\ \alpha_{2n} \\ \vdots \\ \alpha_{mn} \end{bmatrix},$$

which are just linear combinations of the column vectors of  $A$ . Obviously, column vectors of  $A$  are vectors of  $V_m$ . Thus, we have the following result:

*The range of an  $m \times n$  matrix is the span of its column vectors.*

**Rank of  $A$**  The rank of a matrix  $A$  is the dimension of the range of  $A$ , i.e. the span of column vectors of  $A$ . This shows that the rank of  $A$  is the maximum number of linearly independent column vectors of  $A$ .

**Example 5.14** Let

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

The range of  $A$  is  $[(3, 2, 1, 0), (1, -1, 1, 1), (2, 0, 1, 2)]$ . The rank of  $A$  is 3, since the three vectors listed are LI.

**Example 5.15** Let

$$A = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}.$$

We have  $R(A) = [(3, 0, 1), (1, 2, -1), (4, 2, 0), (0, 0, 0)] = [(3, 0, 1), (1, 2, -1), (4, 2, 0)] = [(3, 0, 1), (1, 2, -1)]$ , since  $(4, 2, 0) = (3, 0, 1) + (1, 2, -1)$ . The rank of  $A$  is 2, because  $(3, 0, 1)$  and  $(1, 2, -1)$  are LI.

**Kernel of  $A$**  The kernel of  $A$  is the set of all vectors  $x \in V_n$  such that  $Ax = 0_{m \times 1}$ .

**Nullity of  $A$**  The nullity of  $A$  is the dimension of the kernel of  $A$ .

**Example 5.16** Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix}.$$

This is a linear transformation from  $V_2$  to  $V_3$ . We have

$$\begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ -x_1 + 2x_2 \\ 3x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus,  $Ax = 0_{3 \times 1}$  implies

$$2x_1 + x_2 = 0, -x_1 + 2x_2 = 0, 3x_1 = 0.$$

This gives  $x_1 = 0 = x_2$ . Therefore, the kernel of  $A$  is  $\{0\}$  and the nullity of  $A$  is 0.

**Example 5.17** Consider the matrix of Example 5.15. We have

$$\begin{bmatrix} 3 & 1 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_1 + x_2 + 4x_3 \\ 2x_2 + 2x_3 \\ x_1 - x_2 \end{bmatrix}.$$

Thus,  $Ax = 0$  implies  $3x_1 + x_2 + 4x_3 = 0$ ,  $2x_2 + 2x_3 = 0$ ,  $x_1 - x_2 = 0$ . This gives  $x_1 = x_2 = -x_3$ . So the kernel of  $A$  is the set of all vectors of the form  $(x_1, x_2, -x_3, x_4) = x_1(1, 1, -1, 0) + x_4(0, 0, 0, 1)$ . It is therefore the span  $[(1, 1, -1, 0), (0, 0, 0, 1)]$ . So the nullity of  $A$  is 2.

In Examples 5.14-5.17 rank + nullity = dimension of  $V_n$ . This theorem, which is true for linear transformations between finite-dimensional vector spaces, is also true for matrices, because of the identification of matrices as linear transformations in the above discussion.

Recall that a linear transformation from  $V_n$  to  $V_n$  is one-one *iff* it is onto. So an  $n \times n$  matrix  $A$  will be a one-one linear transformation *iff* its range is the entire space, i.e. *iff* its rank is  $n$ . This means that the maximum number of linearly independent column vectors is  $n$ . In other words, the column vectors of  $A$  are LI. Since the foregoing argument is reversible, we have the following theorem.

**5.5.2 Theorem** *A square matrix is nonsingular iff its column vectors are LI.*

**Example 5.18** Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix}.$$

The range of  $A$  is  $[(1, 1, -1), (0, 2, 1), (1, 3, 0)]$ . For the kernel of  $A$ , consider

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives  $x_1 + x_3 = 0$ ,  $x_1 + 2x_2 + 3x_3 = 0$ ,  $-x_1 + x_2 = 0$ . Solving these equations, we get  $x_1 = -x_3$ ,  $x_1 = x_2$ . So the kernel of  $A$  is  $[(1, 1, -1)]$ . Therefore, the nullity of  $A$  is 1. So the rank of  $A$  is  $3 - 1 = 2$ .

The same result is obtained by inspecting the range of  $A$ . The range of  $A$  is  $[(1, 1, -1), (0, 2, 1)]$ , because  $(1, 3, 0) = (1, 1, -1) + (0, 2, 1)$ . Hence, the matrix  $A$  is singular.

**Example 5.19** Let

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

The range of  $A$  is  $[(2, 1, 1), (1, 2, -1), (-1, 0, 1)]$ . The rank of  $A$  is 3, because we can easily check that these three vectors are LI. Hence,  $A$  is nonsingular. This can also be arrived at by showing that the kernel of  $A$  is  $\{0\}$ .

**Example 5.20** Prove that  $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$  is nonsingular, and find its inverse.

The column vectors  $(1, 0)^T$  and  $(2, 1)^T$  are LI. Therefore, the matrix

is nonsingular. Assume that the inverse of  $A$  is  $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ . Then

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

i.e. 
$$\begin{bmatrix} \alpha + 2\gamma & \beta + 2\delta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So  $\alpha + 2\gamma = 1$ ,  $\beta + 2\delta = 0$ ,  $\gamma = 0$ ,  $\delta = 1$ , which gives  $\alpha = 1$ ,  $\beta = -2$ ,  $\gamma = 0$ ,  $\delta = 1$ . It is easily seen that

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence,

$$A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

**Example 5.21** Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}.$$

The reader can check that the column vectors of  $A$  are LI and so  $A$  is nonsingular. To find the inverse, we solve the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We get nine equations in nine unknowns. Solving, we get

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$$

Alternate methods (which are more powerful) of finding the inverse of a nonsingular matrix will be described in § 5.9.

### Problem Set 5.5

1. Find the range, kernel, rank, and nullity of the following matrices :

(a)  $\begin{bmatrix} 1 & 3 & 2 \\ -1 & 7 & 2 \\ 1 & 0 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & -1 & 2 \\ 3 & -2 & 5 \end{bmatrix}$  (c)  $\begin{bmatrix} 2 & 0 & 1 \\ 7 & 1 & 2 \\ 3 & -1 & 1 \end{bmatrix}$

$$\begin{aligned}
 & \text{(d)} \begin{bmatrix} 2 & 3 & 1 & 2 & 0 \\ 0 & 3 & -1 & 2 & 1 \\ 1 & -3 & 2 & 4 & 3 \\ 2 & 3 & 0 & 3 & 0 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & 3 & -1 & 1 \\ 1 & 5 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
 & \text{(f)} \begin{bmatrix} -1 & 1 & 1 \\ 3 & 1 & -1 \\ 2 & 2 & 1 \end{bmatrix}
 \end{aligned}$$

2. Prove that the following matrices are nonsingular and find their inverses :

$$\begin{aligned}
 & \text{(a)} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} -1 & -1 & 2 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
 & \text{(d)} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 3 & 1 & -1 \\ 1 & 1 & -1 & 0 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}
 \end{aligned}$$

3. Find the values of  $\alpha$  and  $\beta$  for which the following matrix is invertible. Find the inverse when it exists

$$\begin{bmatrix} \alpha & \beta & 0 \\ 0 & \alpha & \beta \\ \beta & 0 & \alpha \end{bmatrix}$$

4. Prove the following :

- If two rows of a matrix are interchanged, then the rank does not change
- If a row of a matrix is multiplied by a nonzero scalar, then the rank does not change.

5. True or false ?

- The range and kernel of a square matrix are of the same dimension.
- There exists a  $7 \times 12$  matrix whose rank is 10.
- If two columns of a matrix are interchanged, then the rank does not change.
- If a column of a matrix is multiplied by a nonzero scalar, then the rank does not change.

- (e) The rank of an  $m \times n$  matrix having  $r$  rows of only zeros is less than or equal to  $m - r$ .

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix}.$$

- (f) The kernel of  $A$  is 2-dimensional.  
 (g) The kernel of  $B$  is 2-dimensional.  
 (h) The ranks of  $A$  and  $B$  are the same

## 5.6 TRANSPOSE OF A MATRIX AND SPECIAL TYPES OF MATRICES

In this article the matrices involved may be considered real or complex.

Let  $A$  be an  $m \times n$  matrix  $(\alpha_{ij})$ . Writing the rows of  $A$  as columns (and therefore the columns of  $A$  as rows), we get a new matrix called the *transpose* of  $A$ , which is denoted by  $A^T$ . Actually, the  $ij$ -th element of  $A^T$  is  $\alpha_{ji}$ , as can be seen from the details of the following entries:

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1j} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2j} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{j1} & \alpha_{j2} & \dots & \alpha_{jj} & \dots & \alpha_{jn} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mj} & \dots & \alpha_{mn} \end{bmatrix}, A^T = \begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{j1} & \dots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{j2} & \dots & \alpha_{m2} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{1j} & \alpha_{2j} & \dots & \alpha_{jj} & \dots & \alpha_{mj} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{jn} & \dots & \alpha_{mn} \end{bmatrix}.$$

Clearly,  $A^T$  is an  $n \times m$  matrix whose  $ij$ -th element is  $\alpha_{ji}$ , the  $ji$ -th element of  $A$ . It follows immediately from the definition of transpose that  $I_n^T = I_n$  and  $0_{m \times n}^T = 0_{n \times m}$ .

**5.6.1 Theorem** If  $A$  and  $B$  are two  $m \times n$  matrices, then

- (a)  $(A + B)^T = A^T + B^T$ ,  
 (b)  $(\alpha A)^T = \alpha A^T$ , and  
 (c)  $(A^T)^T = A$ .

The proof is straightforward and left to the reader.

**5.6.2 Theorem** If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times p$  matrix, then  $(AB)^T = B^T A^T$ .

*Proof:* First, note that  $AB$  is defined and is an  $m \times p$  matrix. So  $(AB)^T$  is a  $p \times m$  matrix. Again,  $A^T$  is an  $n \times m$  matrix and  $B^T$  is a  $p \times n$  matrix. Therefore, we have  $p \times m$  matrices on both sides of the equality in the theorem and so the equality is meaningful.

To prove that the equality holds, it suffices to prove that the  $ij$ -th element of  $(AB)^T$  is equal to the  $ij$ -th element of  $B^T A^T$ . Let  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{n \times p}$ . Then  $A^T = (a'_{ij})_{n \times m}$  and  $B^T = (b'_{ij})_{p \times n}$ , where  $a'_{ij} = a_{ji}$  and  $b'_{ij} = b_{ji}$ . The  $ij$ -th element of  $B^T A^T$  is

$$\sum_{k=1}^n b'_{ik} a'_{kj} = \sum_{k=1}^n b_{ki} a_{jk} = \sum_{k=1}^n a_{jk} b_{ki},$$

which is the  $ji$ -th element of  $AB$  and hence the  $ij$ -th element of  $(AB)^T$ . Thus,  $(AB)^T = B^T A^T$ . ■

**5.6.3 Theorem** *If  $A$  is a nonsingular square matrix, then  $A^T$  is also nonsingular and  $(A^T)^{-1} = (A^{-1})^T$ .*

*Proof:* Since  $A$  is nonsingular, there exists a matrix  $B$  such that  $AB = I = BA$ . Therefore,

$$(AB)^T = I^T = (BA)^T.$$

This gives  $B^T A^T = I = A^T B^T$ . Therefore,  $A^T$  has the inverse, namely,  $B^T$ , and  $(A^T)^{-1} = B^T = (A^{-1})^T$ , because  $B = A^{-1}$ . ■

**5.6.4 Corollary** *The columns of a square matrix  $A$  are LI iff its rows are LI.*

*Proof:* Columns of  $A$  are LI  $\Leftrightarrow A$  is nonsingular (Theorem 5.5.2)  $\Leftrightarrow A^T$  is nonsingular (Theorem 5.6.3)  $\Leftrightarrow$  columns of  $A^T$  are LI (Theorem 5.5.2)  $\Leftrightarrow$  rows of  $A$  are LI (definition of  $A^T$ ). ■

We shall prove in Theorem 5.7.5 that in any matrix, not necessarily square, the maximum number of linearly independent columns is the same as the maximum number of linearly independent rows.

Before we conclude this article let us familiarize ourselves with certain special types of matrices.

A square matrix of the form

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \lambda_n \end{bmatrix}$$

is called a *diagonal matrix*. In other words,  $A = (a_{ij})_{n \times n}$  is a diagonal matrix if  $a_{ij} = 0$  whenever  $i \neq j$ , i.e.  $a_{ij} = \lambda_i \delta_{ij}$ .

A square matrix of the form



$$\begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda \end{bmatrix} = \lambda I$$

is called a *scalar matrix*. In other words,  $A = (\alpha_{ij})_{n \times n}$  is a scalar matrix if  $\alpha_{ij} = \lambda \delta_{ij}$  for a fixed scalar  $\lambda$ .

A square matrix of the form

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ 0 & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & \dots & \alpha_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \alpha_{nn} \end{bmatrix}$$

is called an *upper triangular matrix*. In other words,  $A = (\alpha_{ij})_{n \times n}$  is an upper triangular matrix if  $\alpha_{ij} = 0$  whenever  $i > j$ .

A square matrix of the form

$$\begin{bmatrix} \alpha_{11} & 0 & 0 & \dots & 0 \\ \alpha_{21} & \alpha_{22} & 0 & \dots & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \dots & \alpha_{nn} \end{bmatrix}$$

is called a *lower triangular matrix*. In other words,  $A = (\alpha_{ij})_{n \times n}$  is a lower triangular matrix if  $\alpha_{ij} = 0$  whenever  $i < j$ .

A square matrix  $A$  is said to be *idempotent* if  $A^2 = A$ .

A square matrix  $A$  is said to be *symmetric* if  $A = A^T$ , i.e.  $\alpha_{ij} = \alpha_{ji}$  for all  $i, j$ .

A square matrix  $A$  is said to be *skew-symmetric* if  $A = -A$ , i.e.  $\alpha_{ij} = -\alpha_{ji}$  for all  $i, j$ .

If  $A = (\alpha_{ij})_{m \times n}$ , the matrix  $B = (\bar{\alpha}_{ij})_{m \times n}$ , where the bar denotes complex conjugation, is called the *conjugate* of  $A$ . It is denoted by  $\bar{A}$ .

The matrix  $(\bar{A})^T$  is called the *transposed conjugate* of  $A$ . It is denoted by  $A^*$ . Obviously,  $(\bar{A})^T = (A^T)^*$ .

A matrix  $A$  is said to be *Hermitian* if  $A = A^*$ , i.e. if  $\alpha_{ij} = \bar{\alpha}_{ji}$ . It is said to be *skew-Hermitian* if  $A = -A^*$ , i.e.  $\alpha_{ij} = -\bar{\alpha}_{ji}$ . For example,

$\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$  and  $\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$  are Hermitian matrices

and  $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$  is a skew-Hermitian matrix.

From these definitions of special matrices, we get the following results :

(i) The product of two diagonal matrices of the same order  $n$  is again a diagonal matrix of order  $n$  and

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & \dots & 0 \\ 0 & \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \mu_1 & 0 & \dots & 0 \\ 0 & \lambda_2 \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \mu_n \end{bmatrix}.$$

(ii) Any two diagonal matrices of the same order commute with each other.

(iii) A diagonal matrix is nonsingular *iff* none of the diagonal entries is zero.

(iv) The inverse of the diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \text{ is } \begin{bmatrix} 1/\lambda_1 & 0 & \dots & 0 \\ 0 & 1/\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/\lambda_n \end{bmatrix},$$

$$\lambda_i \neq 0, \quad i = 1, 2, \dots, n.$$

(v) The transpose of an upper (lower) triangular matrix is lower (upper) triangular.

(vi) In a skew-symmetric matrix all the diagonal entries are zero.

(vii) In a Hermitian matrix all the diagonal elements are real.

(viii) Multiplication of a matrix  $A$  by the scalar matrix  $\lambda I$  is equivalent to multiplication of  $A$  by the scalar  $\lambda$ .

### Problem Set 5.6

1. Prove Theorem 5.6.1.

2.  $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 3 & 1 & -2 \end{bmatrix}$ ,  $B = \begin{bmatrix} 1 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ ,  $C = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ -2 & 3 \end{bmatrix}$ .

$$D = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}, E = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

For these matrices, evaluate the following :

- (a)  $A^T$  (b)  $B^T$  (c)  $AB^T$  (d)  $(CB)^T$  (e)  $(AC)^T$  (f)  $BA^T$   
 (g)  $C^T B^T$  (h)  $B + D^T$  (i)  $(C + D)^T$ .
3. For the matrices in Problem 2, verify the following :  
 (a)  $(A + E)^T = A^T + E^T$  (b)  $\{D^T(C + B^T)\}^T = (C^T + B)D$   
 (c)  $(3C + 5D)^T = 3C^T + 5D^T$  (d)  $(AC)^T = C^T A^T$   
 (e)  $(CD^T)^T = DC^T$ .
4. For  $m \times n$  matrices  $A$ ,  $B$ , and  $C$ , prove that  
 (a)  $(AB^T)^T = BA^T$  (b)  $(A^T B)^T = B^T A$   
 (c)  $((A + B)C^T)^T = C(A^T + B^T)$  (d)  $((A + B)^T C)^T = C^T(A + B)$
5. Determine  $A^*$  for the matrix  $A$  in the following :
- (a)  $\begin{bmatrix} 1 + i & 3 - i \\ -1 - i & 2 - i \end{bmatrix}$  (b)  $\begin{bmatrix} 1 - i & 2 - 3i & 1 + i \\ 2 + i & 1 + 4i & 3 - 2i \end{bmatrix}$
- (c)  $\begin{bmatrix} 2 + 3i & 2 - i \\ 1 - 2i & 2 - 2i \\ 3 + 4i & 2 + i \end{bmatrix}$  (d)  $\begin{bmatrix} 1 & 1 - i & 2i - 1 \\ 2 - i & i & -i \\ 1 + i & 2 + i & 3 + 2i \end{bmatrix}$ .
6. If  $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$  and  $X = (x, y, z)$ , evaluate  $XAX^T$ .
7. If  $A$  is an  $m \times n$  matrix with complex entries such that  $AA^* = 0$ , prove that  $A = \bar{A} = 0$ .
8. If  $A$  and  $B$  are square matrices of the same order and  $A$  is symmetric, prove that  $B^T A B$  is also symmetric.
9. (a) Prove that a triangular matrix whose leading diagonal entries are all zeros is nilpotent.  
 (b) Determine a matrix that is both upper and lower triangular.
10. Let  $A$  be a square matrix. Then prove that  
 (a)  $A + A^*$  is Hermitian.  
 (b)  $A - A^*$  is skew-Hermitian.
11. (a) Prove that every square matrix can be expressed as the sum of a Hermitian and a skew-Hermitian matrix.

- (b) Prove that every square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix.
12. Prove that the set of all  $n \times n$  diagonal matrices is a subspace of  $M_{n,n}$ .
13. Prove that the set of all  $n \times n$  invertible diagonal matrices is a group under multiplication.
14. True or false ?
- (a) The set of all  $n \times n$  diagonal matrices is a group under multiplication.
- (b) If  $A$  is involutory, then its transpose is also involutory. ( $A$  is involutory means  $A^2 = I$ )
- (c) The matrix  $\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 1 \end{bmatrix}$  is skew-symmetric.
- (d) Every symmetric matrix is Hermitian.
- (e) A nonzero idempotent matrix is not nilpotent.
- (f) An idempotent matrix  $A$  is singular unless  $A = I$ .

## 5.7 ELEMENTARY ROW OPERATIONS

In this article we shall study certain operations on a matrix that help us to determine its rank. First, we start with the familiar method of solving a system of simultaneous linear equations in three unknowns.

Two systems of (simultaneous) linear equations are said to be *equivalent* (in symbols,  $\Leftrightarrow$ ) if they have the same set of solutions.

Let us solve the system

$$\begin{aligned} 2x - 3y + z &= -1 \\ 3x + 0y + z &= 6 \\ x + 2y - 2z &= -1 \end{aligned} \quad (\text{A})$$

Interchanging the first and third equations, we get the system

$$\begin{aligned} x + 2y - 2z &= -1 & (\text{E1}) \\ 3x + 0y + z &= 6 & (\text{E2}) \\ 2x - 3y + z &= -1 & (\text{E3}) \end{aligned}$$

$\Leftrightarrow$

$$(\text{E1}) : x + 2y - 2z = -1 \quad (\text{E4})$$

$$(\text{E1}) \times (-3) + (\text{E2}) : 0x - 6y + 7z = 9 \quad (\text{E5})$$

$$(\text{E1}) \times (-2) + (\text{E3}) : 0x - 7y + 5z = 1 \quad (\text{E6})$$

$\Leftrightarrow$

$$(\text{E4}) : x + 2y - 2z = -1 \quad (\text{E7})$$

$$(\text{E5}) \times \left(-\frac{1}{6}\right) : 0x + y - \frac{7}{6}z = -\frac{3}{2} \quad (\text{E8})$$

$$(\text{E6}) : 0x - 7y + 5z = 1 \quad (\text{E9})$$

$$\begin{array}{ll}
 \Leftrightarrow & \begin{array}{ll} \text{(E7)} : & x + 2y - 2z = -1 & \text{(E10)} \\ \text{(E8)} : & 0x + y - \frac{7}{6}z = -\frac{3}{2} & \text{(E11)} \\ \text{(E8)} \times (7) + \text{(E9)} & 0x + 0y - \frac{1}{6}z = -\frac{1}{2} & \text{(E12)} \end{array} \\
 \Leftrightarrow & \begin{array}{ll} \text{(E10)} : & x + 2y - 2z = -1 & \text{(E13)} \\ \text{(E11)} & 0x + y - \frac{7}{6}z = -\frac{3}{2} & \text{(E14)} \\ \text{(E12)} \times (-\frac{6}{1}) & 0x + 0y + z = 3 & \text{(E15)} \end{array} \\
 \Leftrightarrow & \begin{array}{ll} \text{(E13)} : & 1x + 2y - 2z = -1 & \text{(E16)} \\ \text{(E15)} \times (\frac{7}{6}) + \text{(E14)} : & 0x + 1y + 0z = 2 & \text{(E17)} \\ \text{(E15)} : & 0x + 0y + 1z = 3 & \text{(E18)} \end{array} \\
 \Leftrightarrow & \begin{array}{ll} \text{(E17)} \times (-2) + \text{(E16)} & 1x + 0y - 2z = -5 & \text{(E19)} \\ \text{(E17)} : & 0x + 1y + 0z = 2 & \text{(E20)} \\ \text{(E18)} & 0x + 0y + 1z = 3 & \text{(E21)} \end{array} \\
 \Leftrightarrow & \begin{array}{ll} \text{(E21)} \times (2) + \text{(E19)} & 1x + 0y + 0z = 1 & \\ \text{(E20)} & 0x + 1y + 0z = 2 & \text{(B)} \\ \text{(E21)} & 0x + 0y + 1z = 3 & \end{array}
 \end{array}$$

which gives  $x = 1$ ,  $y = 2$ , and  $z = 3$ .

It can be seen that throughout we have worked with only the coefficients in the equations and the numbers on the right-hand side. The presence of the symbols  $x$ ,  $y$ , and  $z$  and the sign of equality does not at all affect the working. Now, deleting the symbols  $x$ ,  $y$ , and  $z$  and the sign of equality from the pattern, which otherwise remains unchanged, we shall write only the numbers involved in the form of a matrix  $A$ . Repeating the foregoing sequence of steps, we get the following sequence of matrices, where the symbol  $\sim$  is an analogue of the symbol  $\Leftrightarrow$ , and the letters  $r_1$ ,  $r_2$ , and  $r_3$  stand respectively for row 1, row 2, and row 3. Further,  $r_i + kr_j$  means 'add  $k$  times the  $j$ -th row vector to the  $i$ -th row vector'. We have

$$\begin{array}{ll}
 \text{matrix } A = \begin{bmatrix} 2 & -3 & 1 & -1 \\ 3 & 0 & 1 & 6 \\ 1 & 2 & -2 & -1 \end{bmatrix} & \begin{array}{l} \sim \\ \text{interchange} \\ r_1 \text{ and } r_3 \end{array} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 3 & 0 & 1 & 6 \\ 2 & -3 & 1 & -1 \end{bmatrix} \\
 r_2 + (-3)r_1 & \begin{array}{l} \sim \\ \text{and} \\ r_3 + (-2)r_1 \end{array} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & -6 & 7 & 9 \\ 0 & -7 & 5 & 1 \end{bmatrix} & r_2 \times (-\frac{1}{6}) \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -\frac{7}{6} & -\frac{3}{2} \\ 0 & -7 & 5 & 1 \end{bmatrix} \\
 r_3 + 7r_2 & \begin{array}{l} \sim \\ r_3 \times (-\frac{1}{15}) \end{array} \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -\frac{7}{6} & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{6} & -\frac{1}{2} \end{bmatrix} & \begin{bmatrix} 1 & 2 & -2 & -1 \\ 0 & 1 & -\frac{7}{6} & -\frac{3}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix}
 \end{array}$$



we shall prove that for any matrix  $A$  the column rank is equal to the row rank; so we can afford to call both of them simply 'rank'. Till then, we shall have to distinguish between the row rank defined in 5.7.2 and the rank (= column rank) defined in § 5.5.

The aforesaid process of obtaining matrix  $B$  from matrix  $A$  is called the process of *row reduction*. In practice we usually aim at getting  $B$  in a standard form called *row-reduced echelon form*.

Consider the following  $7 \times 11$  matrix and the 'stairs' or 'steps' marked therein :

Such a matrix is said to be in row-reduced echelon form. We shall now give its precise definition.

**5.7.4 Definition** A matrix is said to be in *row-reduced echelon form* if it satisfies the following criteria :

- (a) The first nonzero entry in each nonzero row is 1.
- (b) If a column contains the first nonzero entry of any row, then every other entry in that column is zero.
- (c) The zero rows (i.e. rows containing only zeros) occur below all the nonzero rows.
- (d) Let there be  $r$  nonzero rows. If the first nonzero entry of the  $i$ -th row occurs in column  $k_i$  ( $i = 1, 2, \dots, r$ ), then  $k_1 < k_2 < \dots < k_r$ .

Draw horizontal and vertical partition lines such that below and left of these lines there are only zeros and such that at the point where the vertical line is followed by a horizontal line there are 1's, namely, the first nonzero entries of nonzero rows. These turning points are called steps. See the foregoing  $7 \times 11$  matrix, where there are six steps.

We now state the main theorem of this article.

**5.7.5 Theorem** (a) *The row rank and column rank of a matrix  $A$  are the same. In other words, the maximum number of linearly independent row vectors is equal to the maximum number of linearly independent column vectors and is equal to the rank of the matrix.*

(b) *The rank of a matrix  $A$  is the number of nonzero rows in its row-reduced echelon form.*

We shall prove this theorem by establishing a succession of lemmas. The sequence of proof can be easily understood from Figure 5.1 :

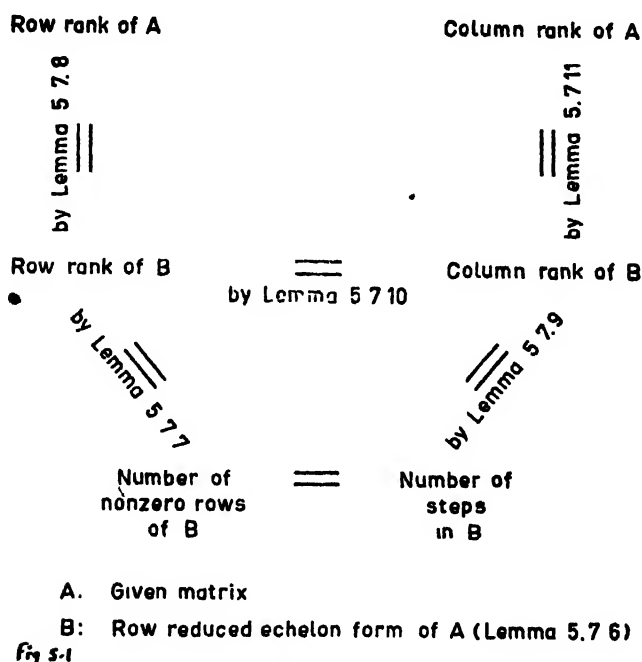


FIGURE 5.1

**5.7.6 Lemma** Every matrix  $A$  is row-equivalent to a row-reduced echelon matrix.

In other words, every matrix can be reduced to the row-reduced echelon form by a finite sequence of elementary row operations.

We shall omit the proof of this lemma, as careful scrutiny of the process of row reduction will convince the reader that this lemma is true.

**5.7.7 Lemma** If a matrix is in the row-reduced echelon form, its row rank is the number of nonzero rows in it.

The proof is left to the reader.

**5.7.8 Lemma** The row rank of a matrix  $A$  is equal to the row rank of the row-reduced echelon matrix  $B$ , obtained from  $A$ .

*Proof:*  $B$  has been obtained from  $A$  by a finite sequence of elementary row operations. We shall show that these row operations do not affect the row rank of  $A$ . It is clear that the row operations of type I and type II do not alter the row rank (see Problem 4, Problem Set 5.5). It is therefore enough to prove that an elementary row operation of type III does not change the maximum number of linearly independent row vectors. Suppose we add  $\alpha$ -times a row vector  $v_i$  to another row vector



$v_2$ . Let the other row vectors be  $v_3, v_4, \dots, v_m$ . Examine the two sets of vectors

$$P = \{v_1, v_2, \dots, v_m\} \text{ and } Q = \{v_1, v_2 + \alpha v_1, v_3, \dots, v_m\}.$$

It can be easily checked that (i) if  $P$  is LI, then  $Q$  is also LI and (ii) if  $P$  is LD, then  $Q$  is LD. This shows that a type III operation does not affect the maximum number of linearly independent row vectors, and so the row rank is unaffected by such an operation (see Problem 9). ■

**5.7.9 Lemma** *If a matrix is in the row-reduced echelon form, then its (column) rank is the number of 'steps' in it.*

*Proof:* Let the number of steps be  $p$ . Every column that occurs before the first step is a zero vector, and every row that occurs after the last step is also a zero vector. So the nonzero column vectors can be considered vectors in  $V_p$ . Therefore, the column rank is less than or equal to  $p$ . On the other hand each step contributes one nonzero column vector. The set of these column vectors is LI. This means the column rank is greater than or equal to  $p$ . Thus, the column rank is  $p$ . ■

**5.7.10 Lemma** *If a matrix is in the row-reduced echelon form, then its (column) rank is equal to its row rank.*

*Proof:* Note that the number of steps in such a matrix is equal to the number of nonzero rows. Using Lemmas 5.7.7 and 5.7.9, we find that its (column) rank is the number of steps in it, i.e. the number of nonzero rows, i.e. its row rank. ■

**5.7.11 Lemma** *The (column) rank of a matrix  $A$  is equal to the (column) rank of the row-reduced echelon matrix  $B$ , obtained from  $A$ .*

*Proof:* To facilitate understanding of this lemma, we shall prove it for a matrix of order  $3 \times 4$ . By using proper notations, the same proof can be extended to any  $m \times n$  matrix.

We have only to prove that the three types of elementary row operations do not affect the column rank. The fact that types I and II operations do not affect the column rank is easy to prove and is left to the reader. To prove that a type III operation does not affect the column rank, consider the matrix

$$P = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \end{bmatrix}.$$

Let us perform a type III operation on  $P$ , namely, let us add  $\alpha$ -times row 3 to row 2. The resulting matrix  $Q$  will be

$$Q = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} + \alpha\alpha_{31} & \alpha_{22} + \alpha\alpha_{32} & \alpha_{23} + \alpha\alpha_{33} & \alpha_{24} + \alpha\alpha_{34} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \end{bmatrix}.$$

Suppose the column rank of  $P$  is 3 (any other value of the rank can be handled analogously). Then in  $P$  there exist three linearly independent column vectors and every four column vectors are LD. We shall prove that the same phenomenon occurs in  $Q$  also.

Let the three linearly independent column vectors of  $P$  be the first three (if they are any other three, they can be brought to the first three positions without affecting the argument). Let us call them  $C_1$ ,  $C_2$ , and  $C_3$ .

If  $F_1$ ,  $F_2$ , and  $F_3$  are the first three column vectors of  $Q$ , then the vector equation  $\beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 = 0$  gives

$$\begin{aligned}\beta_1 \alpha_{11} + \beta_2 \alpha_{12} + \beta_3 \alpha_{13} &= 0 \\ \beta_1(\alpha_{21} + \alpha \alpha_{31}) + \beta_2(\alpha_{22} + \alpha \alpha_{32}) + \beta_3(\alpha_{23} + \alpha \alpha_{33}) &= 0 \\ \beta_1 \alpha_{31} + \beta_2 \alpha_{32} + \beta_3 \alpha_{33} &= 0.\end{aligned}$$

Adding  $(-\alpha)$ -times the third equation to the second equation, we get the equivalent system

$$\begin{aligned}\beta_1 \alpha_{11} + \beta_2 \alpha_{12} + \beta_3 \alpha_{13} &= 0 \\ \beta_1 \alpha_{21} + \beta_2 \alpha_{22} + \beta_3 \alpha_{23} &= 0 \\ \beta_1 \alpha_{31} + \beta_2 \alpha_{32} + \beta_3 \alpha_{33} &= 0.\end{aligned}$$

This system is nothing but the vector equation  $\beta_1 C_1 + \beta_2 C_2 + \beta_3 C_3 = 0$ . Since  $C_1$ ,  $C_2$ , and  $C_3$  are LI, we get  $\beta_1 = 0 = \beta_2 = \beta_3$ . Thus,  $F_1$ ,  $F_2$ , and  $F_3$  are LI.

Again, we prove that when  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  are LD, then automatically  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$  are also LD.

The vector equation  $\beta_1 F_1 + \beta_2 F_2 + \beta_3 F_3 + \beta_4 F_4 = 0$  gives

$$\begin{aligned}\beta_1 \alpha_{11} + \beta_2 \alpha_{12} + \beta_3 \alpha_{13} + \beta_4 \alpha_{14} &= 0 \\ \beta_1(\alpha_{21} + \alpha \alpha_{31}) + \beta_2(\alpha_{22} + \alpha \alpha_{32}) + \beta_3(\alpha_{23} + \alpha \alpha_{33}) + \beta_4(\alpha_{24} + \alpha \alpha_{34}) &= 0 \\ \beta_1 \alpha_{31} + \beta_2 \alpha_{32} + \beta_3 \alpha_{33} + \beta_4 \alpha_{34} &= 0.\end{aligned}$$

Adding  $(-\alpha)$ -times the third equation to the second equation, we get an equivalent system which is nothing but the vector equation  $\beta_1 C_1 + \beta_2 C_2 + \beta_3 C_3 + \beta_4 C_4 = 0$ .

Since  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  are LD, at least one  $\beta$  is not zero. Hence,  $F_1$ ,  $F_2$ ,  $F_3$ , and  $F_4$  are LD.

Thus, we see that the type III operation does not affect the column rank of matrix  $P$ . ■

*Proof of Theorem 5.7.5:* (a) Let  $B$  be the row-reduced echelon form of  $A$ . Then

$$\begin{aligned}\text{row rank of } A &= \text{row rank of } B \text{ (Lemma 5.7.8)} \\ &= \text{column rank of } B \text{ (Lemma 5.7.10)} \\ &= \text{column rank of } A \text{ (Lemma 5.7.11)}.\end{aligned}$$

$$\begin{aligned}\text{(b) Rank of } A &= \text{column rank of } A \text{ (Remark 5.7.3)} \\ &= \text{row rank of } A \text{ (part (a) of this theorem)} \\ &= \text{row rank of } B \text{ (Lemma 5.7.8)} \\ &= \text{number of nonzero rows in } B \text{ (Lemma 5.7.7).} \quad \blacksquare\end{aligned}$$

**5.7.12 Corollary** An  $n \times n$  matrix  $A$  is nonsingular iff its row-reduced echelon form is  $I_n$ .

*Proof:* Let the row-reduced echelon form be  $I_n$ . Clearly, its column vectors are LI. So the rank of  $I_n$  is  $n$ . Hence, by Lemma 5.7.11, the column rank of  $A$  is  $n$ , i.e. column vectors of  $A$  are LI. Therefore, by Theorem 5.5.2,  $A$  is nonsingular.

Conversely, let  $A$  be nonsingular. So the rank of  $A$  is  $n$ . Therefore, the row-reduced echelon form  $B$  of  $A$  will have  $n$  linearly independent column vectors (Lemma 5.7.11). Since the number of columns in  $B$  is  $n$ , it follows that  $B = I_n$ . ■

**Example 5.22** Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

by reducing it to row-reduced echelon form

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix} \xrightarrow{\substack{r_2 + r_1, \\ r_3 - 2r_1, \\ r_4 - r_1}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix} \\
 &\xrightarrow{\substack{\text{inter-} \\ \text{changing} \\ r_2 \text{ and } r_4}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix} \xrightarrow{r_2 \times (-1)} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix} \\
 &\xrightarrow{\substack{r_3 + 3r_2, \\ r_4 - 5r_2}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix} \xrightarrow{r_4 + 9r_3} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 &\xrightarrow{r_3 \times (-1)} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\substack{r_2 + 2r_3, \\ r_1 + r_3}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$\sim \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ r_1 - 2r_2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}$$

Hence, the rank of  $A$  is 3.

Recall the explanation at the beginning of this article. The row reduction process is just an abstract imitation of the elimination method we adopt in solving simultaneous linear equations, except that we now do it in an organised way. The row reduction worked out in Example 5.22 is therefore also the method of solving the system

$$\begin{aligned} x + 2y - z &= 0 \\ -x + 3y &= -4 \\ 2x + y + 3z &= -2 \\ x + y + z &= -1, \end{aligned}$$

giving the solution  $x = 1, y = -1, z = -1$ .

This takes us to the subject of the solution of a system of linear equations, which we shall deal with in § 5.8.

### Problem Set 5.7

1. Reduce the following matrices to the row-reduced echelon form :

(a)  $\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}$

(b)  $\begin{bmatrix} 3 & 2 & 3 & 1 \\ 4 & 3 & 5 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{bmatrix}$

(c)  $\begin{bmatrix} 1 & 2 & 3 & 4 & -1 \\ 4 & 1 & 5 & -6 & 10 \\ 2 & 0 & 2 & -2 & 6 \end{bmatrix}$

(d)  $\begin{bmatrix} 1 & 3 & 2 & 0 \\ 2 & 0 & 1 & 1 \\ 1 & 2 & 3 & 0 \\ 3 & -1 & 0 & 1 \end{bmatrix}$

(e)  $\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix}$

(f)  $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 2 \\ 0 & 2 & 1 & -1 \end{bmatrix}$

(g)  $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 2 & 3 \\ 4 & 0 & 1 \\ 1 & 5 & 3 \end{bmatrix}$

(h)  $\begin{bmatrix} 0 & 6 & 6 & 1 \\ -8 & 7 & 2 & 3 \\ -3 & 2 & 1 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix}$

$$(i) \begin{bmatrix} 1 & 0 & 7 & 9 \\ 5 & 2 & 2 & 10 \\ 3 & -2 & 3 & 11 \\ 2 & -1 & 3 & 8 \end{bmatrix}$$

2. Determine the column rank of the matrices in Problem 1 by two different methods.
3. Determine the row rank of the matrices in Problem 1 by two different methods.
4. Determine which of the square matrices in Problem 1 are nonsingular and in each case find the inverse.
5. Solve the following systems of linear equations by using the row-reduction method :
  - (a) 
$$\begin{aligned} 2x - 3y &= 1 \\ 2x - y + z &= 2 \\ 3x + y - 2z &= 1 \end{aligned}$$
  - (b) 
$$\begin{aligned} y - 2z &= 3 \\ 3x + z &= 4 \\ x + y + z &= 1 \end{aligned}$$
  - (c) 
$$\begin{aligned} x - y + z &= 0 \\ 2x + y - 3z &= 1 \\ -x + y + 2z &= -1 \end{aligned}$$
  - (d) 
$$\begin{aligned} x - y + 3z &= 1 \\ 2x + y - z &= 2 \\ 3x - y + 2z &= 2 \end{aligned}$$
  - (e) 
$$\begin{aligned} x + y - 2z &= 3 \\ 3x + y - z &= 8 \\ 2x - y + z &= 0. \end{aligned}$$
6. Prove that ' $\sim$ ' is an equivalence relation.
7. In the  $7 \times 11$  matrix on p. 187 find the numbers  $k_1, k_2, \dots, k_r$  of Definition 5.7.4.
8. Prove Lemma 5.7.7.
9. Write down the details of the proof that the row operation of type III does not affect the row rank of the matrix.
10. True or false ?
  - (a) For any matrix  $A$ , the ranks of  $A$  and  $A^T$  are the same.
  - (b) The row-reduced echelon form of a diagonal matrix  $A$  is  $A$  itself.
  - (c) Every upper triangular matrix is row equivalent to a diagonal matrix.
  - (d) There exists a  $12 \times 7$  matrix of rank 10.
  - (e) The row-reduced echelon form of a symmetric matrix is also symmetric.
  - (f) The type III operation performed on the columns of a matrix leaves its rank unchanged.

[illegible][illegible]
$$Ax = 0,$$
$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix},$$

This operator equation is to be solved for  $x$ . The matrix  $A$  is called the *coefficient matrix* of the system. The matrix obtained by adjoining the column vector  $b$ , at the end, to the matrix  $A$ , is called the *augmented matrix* of the system (NH) and is denoted by  $(A, b)$ . As usual, we consider  $A$  a linear transformation from  $V_n$  to  $V_m$ . By Theorem 4.8.1, we have

**Thus, we have proved the following theorem.**

(b) (Uniqueness) *If the system (NH) has a solution, then the solution is unique iff the rank of  $A$  is equal to  $n$ .*

This theorem and Theorem 4.8.1, properly interpreted for our systems of equations (NH) and (H), give the following comprehensive theorem.

**5.8.2 Theorem** Consider the systems of equations (NH) and (H). Let the rank of  $A$  be  $r$ . Then

- (a) (NH) has a solution iff  $(A, b)$  has rank  $r$ .
- (b) If  $r = m$ , then (NH) always has a solution, whatever may be  $b \in V_m$ .
- (c) If  $r = m = n$ , then (NH) has a unique solution, whatever may be  $b \in V_m$ ; and further (H) has a unique solution, namely, the trivial solution.
- (d) If  $r = m < n$ , whatever may be  $b \in V_m$ , (NH) as well as (H) have an infinite number of solutions. In fact,  $r$  of the unknowns can be determined in terms of the remaining  $(n - r)$  unknowns, whose values can be arbitrarily chosen.
- (e) In the cases (i)  $r < m = n$ , (ii)  $r < m < n$ , and (iii)  $r < n < m$ , if (NH) has a solution, then there is an infinite number of solutions. In fact,  $r$  of the unknowns can be determined in terms of the remaining  $(n - r)$  unknowns, whose values can be arbitrarily chosen. Further, (H) has an infinite number of solutions.
- (f) In the case  $r = n < m$ , (H) has a unique solution, namely, the trivial solution, and if (NH) has a solution, then that solution is unique.
- (g) If  $m = n$ , (H) has a nontrivial solution iff  $A$  is singular.

*Proof:* Parts (a) and (f) are only restatements of Theorem 5.8.1. They are included here for completeness. Part (b) follows, since  $r = m$  implies the range of  $A$  is  $V_m$ . Part (c) follows from part (b) and Theorem 5.8.1 (b). Now we have to prove only parts (d), (e), and (g).

To prove part (d), first note that, since  $r = m$ , the range of  $A$  is  $V_m$  and so every  $b \in V_m$  has an  $A$  pre-image in  $V_n$ . The kernel of  $A$  has the dimension  $n - r > 0$ . So the kernel  $K$ , being a subspace, has an infinite number of vectors in it. By Theorem 4.8.1, it follows that the solution set of (NH) is a translate of  $K$ . So it also has an infinite number of vectors. The fact that  $r$  unknowns can be determined in terms of the remaining  $(n - r)$  unknowns can be seen from the row reduction process of  $A$  (see Example 5.23).

To prove part (e), note that the proof is the same as that of part (d), except that, first, we should know whether there exists a solution. Once the solution exists, the rest of the argument is the same.

Finally, part (g) follows from parts (e(i)) and (c) once we recall that an  $n \times n$  square matrix is singular iff its rank is less than  $n$ . ■

The result of this theorem can be presented as in Table 5.1.

Let the rank of matrix  $A$  be  $r$  and that of the augmented matrix  $(A, b)$  be  $r_1$ . Further, let the rank of the kernel be  $k = n - r$ . Obviously,  $r < r_1$ ,  $r < m$ ,  $r < n$ . (Numbers within brackets refer to the parts of Theorem 5.8.2.)

TABLE 5.1

$r \neq r_1$		The system has no solution (a)	
$r = r_1$		The system has a solution and the following happens. (a)	
		Homogeneous system (H) $Ax = 0$	Nonhomogeneous system (NH) $Ax = b, b \neq 0$
$m = n$	$r = n, k = 0$	has only the trivial solution (c)	has the unique solution (c)
	$r < n, k = n - r \geq 1$	has infinitely many solutions. Solution space is of dimension $k = n - r$ (e(i))	has infinitely many solutions. Solution set is a linear variety whose base space is of dimension $k$ (e(i))
		We can find $r$ unknowns in terms of the remaining $k (= n - r)$ unknowns, chosen arbitrarily (e)	
$m > n$	$r = n, k = 0$	has only the trivial solution (f)	has the unique solution (f)
	$r < n, k = n - r \geq 1$	has infinitely many solutions. Solution space is of dimension $k (= n - r)$ (e(iii))	has infinitely many solutions. Solution set is a linear variety whose base space is of dimension $k$ (e(iii))
		We can find $r$ unknowns in terms of the remaining $k (= n - r)$ unknowns, chosen arbitrarily (e)	
$m < n$	$r = m, k = n - m$	has infinitely many solutions. Solution space is of dimension $k (= n - m)$ (d)	has infinitely many solutions. Solution set is a linear variety whose base space is of dimension $k$ (d)
		We can find $r (= m)$ unknowns in terms of the remaining $k (= n - m)$ unknowns, chosen arbitrarily (d)	
	$r < m, k = n - r$	has infinitely many solutions. Solution space is of dimension $k (= n - r)$ (e(ii))	has infinitely many solutions. Solution set is a linear variety whose base space is of dimension $k$ (e(ii))
		We can find $r$ unknowns in terms of the remaining $k (= n - r)$ unknowns, chosen arbitrarily (e)	



**Example 5.23** Consider the system

$$\begin{aligned} 2x_1 + x_3 - x_4 + x_5 &= 2 \\ x_1 + x_3 - x_4 + x_5 &= 1 \\ 12x_1 + 2x_2 + 8x_3 + 2x_5 &= 12. \end{aligned}$$

The augmented matrix  $(A, b)$

$$\begin{aligned} &= \left[ \begin{array}{cccccc} 2 & 0 & 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & -1 & 1 & 1 \\ 12 & 2 & 8 & 0 & 2 & 12 \end{array} \right] \begin{array}{l} \sim \\ r_1 - 2r_2, \\ r_3 - 2r_2 \end{array} \left[ \begin{array}{cccccc} 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 2 & -4 & 12 & -10 & 0 \end{array} \right] \\ &\quad \begin{array}{l} \sim \\ \text{interchange} \\ r_1 \text{ and } r_2, \\ r_3 \times \frac{1}{2} \end{array} \left[ \begin{array}{cccccc} 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 6 & -5 & 0 \end{array} \right] \\ &\quad \begin{array}{l} \sim \\ r_1 + r_2, \\ r_3 - 2r_2 \end{array} \left[ \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & -3 & 0 \end{array} \right] \\ &\quad \begin{array}{l} \sim \\ \text{interchange} \\ r_2 \text{ and } r_3 \end{array} \left[ \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array} \right] \\ &\quad \begin{array}{l} \sim \\ r_3 \times (-1) \end{array} \left[ \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & -3 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array} \right]. \end{aligned}$$

This shows

$$\begin{aligned} x_1 &= 1 \\ x_2 + 4x_4 - 3x_5 &= 0 \\ x_3 - x_4 + x_5 &= 0. \end{aligned}$$

So

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -4x_4 + 3x_5 \\ x_3 &= x_4 - x_5. \end{aligned}$$

Rank of the coefficient matrix  $A = 3 =$  rank of the augmented matrix.  $5 - 3 = 2$  unknowns can be arbitrarily chosen. They are  $x_4$  and  $x_5$ . Three unknowns  $x_1$ ,  $x_2$ , and  $x_3$  are determined in terms of  $x_4$  and  $x_5$ . There is an infinite number of solutions. This is case (d) of Theorem 5.8.2.

The set of solutions can be written as

$$\begin{aligned} &\{(1, -4x_4 + 3x_5, x_4 - x_5, x_4, x_5) \mid x_4, x_5 \text{ are arbitrary scalars}\} \\ &= \{(1, 0, 0, 0, 0) + x_4(0, -4, 1, 1, 0) + x_5(0, 3, -1, 0, 1) \mid x_4, x_5 \\ &\quad \text{are arbitrary scalars}\} \end{aligned}$$

$$= (1, 0, 0, 0, 0) + [(0, -4, 1, 1, 0), (0, 3, -1, 0, 1)] .$$

It is a linear variety. Here  $(1, 0, 0, 0, 0)$  is one particular solution of the system and  $[(0, -4, 1, 1, 0), (0, 3, -1, 0, 1)]$  is the kernel of the coefficient matrix  $A$  of the system.

**Example 5 24** Consider the system

$$\begin{aligned} x_1 + 2x_2 + 4x_3 + x_4 &= 4 \\ 2x_1 - x_3 - 3x_4 &= 4 \\ x_1 - 2x_2 - x_3 &= 0 \\ 3x_1 + x_2 - x_3 - 5x_4 &= 5 . \end{aligned}$$

The augmented matrix  $(A, b)$

$$= \left[ \begin{array}{ccccc} 1 & 2 & 4 & 1 & 4 \\ 2 & 0 & -1 & -3 & 4 \\ 1 & -2 & -1 & 0 & 0 \\ 3 & 1 & -1 & -5 & 5 \end{array} \right] \begin{array}{l} \sim \\ r_2 - 2r_1, \\ r_3 - r_1, \\ r_4 - 3r_1 \end{array} \left[ \begin{array}{ccccc} 1 & 2 & 4 & 1 & 4 \\ 0 & -4 & -9 & -5 & -4 \\ 0 & -4 & -5 & -1 & -4 \\ 0 & -5 & -13 & -8 & -7 \end{array} \right]$$

$$r_2 \times (-\frac{1}{4}) \quad \left[ \begin{array}{ccccc} 1 & 2 & 4 & 1 & 4 \\ 0 & 1 & \frac{9}{4} & \frac{5}{4} & 1 \\ 0 & -4 & -5 & -1 & -4 \\ 0 & -5 & -13 & -8 & -7 \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_1 - 2r_2, \\ r_3 + 4r_2, \\ r_4 + 5r_2 \end{array} \quad \left[ \begin{array}{ccccc} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 2 \\ 0 & 1 & \frac{9}{4} & \frac{5}{4} & 1 \\ 0 & 0 & 4 & 4 & 0 \\ 0 & 0 & -\frac{7}{4} & -\frac{7}{4} & -2 \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_3 \times \frac{1}{4}, \\ r_4 \times (-\frac{4}{7}) \end{array} \quad \left[ \begin{array}{ccccc} 1 & 0 & -\frac{1}{2} & -\frac{3}{2} & 2 \\ 0 & 1 & \frac{9}{4} & \frac{5}{4} & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & \frac{8}{7} \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_1 + \frac{1}{2}r_3, \\ r_2 - \frac{9}{4}r_3, \\ r_4 - r_3 \end{array} \quad \left[ \begin{array}{ccccc} 1 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{8}{7} \end{array} \right]$$

The last row shows that  $0x_1 + 0x_2 + 0x_3 + 0x_4 = 8/7$ , which is

absurd. Otherwise also it shows that the rank of  $A$  is 3 and the rank of  $(A, b)$  is 4. (Why?) So the system cannot have a solution. In other words, the equations are inconsistent.

### Problem Set 5.8

1. Determine whether the following systems of linear equations are consistent. Discuss completely the solution in the case of consistent systems.

(a) $x_1 - x_2 + 2x_3 + 3x_4 = 1$	(b) $x_1 + 2x_2 + 4x_3 + x_4 = 4$
$2x_1 + 2x_2 + 2x_4 = 1$	$2x_1 - x_3 + 3x_4 = 4$
$4x_1 + x_2 - x_3 - x_4 = 1$	$x_1 - 2x_2 - x_3 = 0$
$x_1 + 2x_2 + 3x_3 = 1$	$3x_1 + x_2 - x_3 - 5x_4 = 7$
(c) $2x_1 + x_3 - x_4 + x_5 = 2$	(d) $x_1 + 2x_2 - x_3 - 2x_4 = 0$
$x_1 + x_3 - x_4 + x_5 = 1$	$2x_1 + 4x_2 + 2x_3 + 4x_4 = 4$
$12x_1 + 2x_2 + 8x_3 + 2x_5 = 12$	$3x_1 + 6x_2 + 3x_3 + 6x_4 = 6$
(e) $x_1 - x_3 = 1$	(f) $x_1 + 2x_3 = 1$
$2x_1 + x_2 + x_3 = 2$	$2x_1 + x_2 + 2x_3 = 1$
$x_2 - x_3 = 3$	$x_2 - 2x_3 = 1$
$x_1 + x_2 + x_3 = 4$	$x_1 + x_2 = 1$
$2x_2 - x_3 = 0$	$x_1 - x_2 + 4x_3 = 1$
(g) $2x_1 + x_2 + x_3 + x_4 = 2$	(h) $x_1 + 3x_2 - 3x_3 + 2x_4 = 1$
$3x_1 - x_2 + x_3 - x_4 = 2$	$4x_1 + x_2 - 2x_3 + x_4 = 1$
$x_1 + 2x_2 - x_3 + x_4 = 1$	$6x_1 + 5x_2 + 10x_3 + 3x_4 = 15$
$6x_1 + 2x_2 + x_3 + x_4 = 5$	$x_1 + 2x_2 + 3x_3 + x_4 = 6$
(i) $x_1 - 2x_2 - x_3 = -1$	(j) $3x_1 + 6x_2 + 3x_3 + 6x_4 = 5$
$2x_1 - x_3 - 3x_4 = 1$	$x_1 + 2x_3 - x_3 - 2x_4 = -1$
$3x_1 + x_2 - x_3 - 5x_4 = 1$	$3x_1 + 6x_2 + x_3 + 2x_4 = 3$
$2x_1 + 3x_3 + x_4 = 0$	$x_1 + 2x_2 + 2x_3 + 4x_4 = 3$
(k) $x_1 - x_3 = 2$	(l) $2x_1 + x_2 + 2x_3 = 1$
$x_1 + x_2 + 2x_3 = 4$	$x_1 + x_2 = 0$
$x_1 + x_2 - 2x_3 = 4$	$x_1 - 2x_2 + 6x_3 = 3$
$x_1 + x_2 + x_3 = 4$	$x_1 - 2x_3 = 1$
$x_1 + 3x_3 - x_3 = 8$	$x_1 - x_2 + 4x_3 = 2$
(m) $x_1 + x_2 - x_3 - 6x_4 + 6x_5 = -19$	(n) $x_1 - 3x_2 + x_3 - x_4 = 7$
$x_1 + 7x_4 - 7x_5 = 28$	$2x_1 + 4x_2 - x_3 + 6x_4 = -6$
$2x_2 - 3x_3 + 18x_4 - 4x_5 = 24$	$2x_1 + x_2 + x_4 = 0$

### 5.9 MATRIX INVERSION

In this article we apply the method of row reduction to find the inverse of a nonsingular matrix. Let  $A$  be an  $n \times n$  matrix, which is nonsingular. Then  $A^{-1}$  exists. Look at the equation  $Ax = b$ , where  $b \neq 0$ . To solve this we can proceed as follows :

$$\begin{aligned}
 &Ax = b. \\
 \text{Therefore,} \quad &A^{-1}(Ax) = A^{-1}b \\
 \text{or} \quad &(A^{-1}A)x = A^{-1}b \\
 \text{or} \quad &I_n x = A^{-1}b \\
 \text{or} \quad &x = A^{-1}b.
 \end{aligned}$$

So, solving the equation  $Ax = b$  is equivalent to finding  $A^{-1}b$ . But the solution of equations can be done by the process of row reduction. Therefore, the method of row reduction will also be adaptable to find  $A^{-1}b$  and consequently  $A^{-1}$ . We can write the equation  $Ax = b$  as

$$Ax = I_n b. \quad (1)$$

$$\begin{aligned}
 \text{Therefore} \quad &A^{-1}Ax = A^{-1}I_n b, \\
 \text{i.e.} \quad &I_n x = A^{-1}b.
 \end{aligned} \quad (2)$$

In the process of reduction we always work from  $A$  and arrive at  $I_n$  (see Corollary 5.7.12). Equations (1) and (2) indicate that, if the same row reduction is applied to  $I_n$ , the identity matrix, then we end up with the matrix  $A^{-1}$  on the right-hand side of (1). Thus, we have the following method of inverting  $A$ .

Perform a sequence of elementary row operations on  $A$  so that it reduces to  $I_n$ . Perform the same sequence of elementary row operations on the matrix  $I_n$ . The resulting matrix is  $A^{-1}$ .

In practice, we perform the sequence of elementary row operations on  $A$  and  $I_n$  simultaneously, keeping them side by side as illustrated in the following example.

**Example 5.25** Invert the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix}.$$

**Note** Whether the matrix is nonsingular or not will also be clear by the process of row reduction. If the final matrix has all its rows nonzero, then the original matrix should be nonsingular. (Why?)

We write the  $3 \times 3$  matrix  $A$  and the matrix  $I_3$  side by side and obtain a  $3 \times 6$  matrix with a vertical line separating the entries of the matrices  $A$  and  $I_3$  as shown below. Then we perform the elementary row operations on this  $3 \times 6$  matrix in such a way that the entries on the left-hand side of the vertical line ultimately form  $I_3$ . The entries on the right-hand side of the vertical line then give the required inverse of  $A$ . The  $3 \times 6$  matrix used here is called the enlarged matrix of  $A$ .

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array} \right].$$

$$\begin{array}{l} \sim \\ r_2 - r_1, \\ r_3 - r_1 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_2 \times (-\frac{1}{2}), \\ r_3 \times (-\frac{1}{2}) \end{array} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 & \frac{1}{2} & 0 & -\frac{1}{2} \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_3 - r_2 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_3 \times (-2) \end{array} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_2 + \frac{1}{2}r_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

$$\begin{array}{l} \sim \\ r_1 - r_2 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right].$$

Hence,

$$A^{-1} = \left[ \begin{array}{ccc} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 1 \end{array} \right].$$

The reader is advised to check that  $AA^{-1}$  is indeed  $I$ , and so also is  $A^{-1}A$ .

### Problem Set 5.9

1. Using the row reduction method, determine which of the square matrices of Problem 1, Problem Set 5.5, are nonsingular and in each of these cases find the inverse.
2. Repeat Problem 1 for the matrices of Problem 2, Problem Set 5.5.
3. Repeat Problem 1 for the square matrices of Problem 1, Problem Set 5.7.

## Chapter 6

# Determinants

### 6.1 DEFINITION

We start with some preliminaries on permutations of the set  $\{1, 2, \dots, n\}$ .

**6.1.1 Definition** An ordered pair  $(p, q)$  of distinct positive integers  $p$  and  $q$  is said to be an *inversion* if  $p > q$ .

For example,  $(5, 3)$  is an inversion but  $(3, 5)$  is not.

**6.1.2 Definition** Given a permutation  $P = (j_1, j_2, \dots, j_n)$  of the set  $\{1, 2, \dots, n\}$ , we define the set

$$\Phi_P = \{(j_1, j_2), (j_1, j_3), \dots, (j_1, j_n), \\ (j_2, j_3), \dots, (j_2, j_n), \\ \dots, \\ (j_{n-1}, j_n)\}.$$

The number of inversions in  $\Phi_P$  is called the *number of inversions in the permutation  $P$* .

**Example 6.1** Let  $P$  be the permutation  $(3, 4, 1, 5, 2)$ . Then  $\Phi_P = \{(3, 4), (3, 1), (3, 5), (3, 2), (4, 1), (4, 5), (4, 2), (1, 5), (1, 2), (5, 2)\}$ . Counting the inversions, we see that the number of inversions in  $P$  is 5.

**6.1.3 Definition** A permutation  $(j_1, j_2, \dots, j_n)$  of the set  $\{1, 2, \dots, n\}$  is said to be an *even (odd) permutation* if the number of inversions in it is even (odd).

**Example 6.2** The permutation  $(3, 4, 1, 5, 2)$  in Example 6.1 is an odd permutation, whereas  $(3, 1, 4, 5, 2)$  is an even permutation.

Now let us take up the definition of a determinant. To each  $n \times n$  matrix  $A$  of numbers we associate a number called the determinant of  $A$  in the following manner. Let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}.$$

Form all products of the type

$$\alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{nj_n},$$

in that order (the ascending order of row indices). Here  $(j_1, j_2, \dots, j_n)$  is a permutation of the set  $\{1, 2, \dots, n\}$  of column indices. Note that each such product contains one (and only one) term from each row and each column of the matrix. To each such product assign a 'plus' sign or a 'minus' sign according as the permutation  $(j_1, j_2, \dots, j_n)$  is even or odd. The algebraic sum of all these signed products is called the determinant of  $A$ . It is denoted by  $\det A$  or  $|A|$  or

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}.$$

This last symbol is itself called a *determinant of order  $n$* . The precise definition is as follows.

**6.1.4 Definition** If

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}, \quad (1)$$

then

$$\det A = \sum \epsilon(j_1, j_2, \dots, j_n) \alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{nj_n}, \quad (2)$$

where the summation is taken over all possible products of the form  $\alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{nj_n}$ , in that order, and

$$\epsilon(j_1, j_2, \dots, j_n) = \begin{cases} +1 & \text{if } (j_1, j_2, \dots, j_n) \text{ is} \\ & \text{an even permutation} \\ -1 & \text{if } (j_1, j_2, \dots, j_n) \text{ is} \\ & \text{an odd permutation.} \end{cases}$$

The right-hand side of Equation (2) is called the *expansion of the determinant of  $A$* .

Note that there are  $n!$  terms in the expansion of  $\det A$ , because there are  $n!$  permutations of the set  $\{1, 2, \dots, n\}$  of column indices, and to each such permutation  $(j_1, j_2, \dots, j_n)$  we get a product  $\alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{nj_n}$ . In fact, the permutation  $(j_1, j_2, \dots, j_n)$  tells us the order in which the column indices are chosen to form the product.

**6.1.5 Remark**  $\det A$  is defined only when  $A$  is a square matrix. If  $A$  is not a square matrix,  $\det A$  is not defined.

**Example 6.3** Let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}.$$

Here the six possible products are  $\alpha_{11}\alpha_{22}\alpha_{33}$ ,  $\alpha_{11}\alpha_{23}\alpha_{32}$ ,  $\alpha_{12}\alpha_{23}\alpha_{31}$ ,  $\alpha_{12}\alpha_{21}\alpha_{33}$ ,  $\alpha_{13}\alpha_{21}\alpha_{33}$ ,  $\alpha_{13}\alpha_{22}\alpha_{31}$ . The signs to be attached to these products are determined as follows: For example, take  $\alpha_{12}\alpha_{21}\alpha_{33}$ . The order of the columns chosen here is  $(2, 1, 3)$ . This is an odd permutation. So the sign to be attached to this product is 'minus'. Doing this for every product, we finally get

$$|A| = \alpha_{11}\alpha_{22}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31} - \alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{13}\alpha_{21}\alpha_{32} - \alpha_{13}\alpha_{22}\alpha_{31}. \quad (3)$$

As a numerical example, we may take

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & 0 \\ -1 & 1 & 2 \end{vmatrix} = 2 \times 4 \times 2 - 2 \times 0 \times 1 + (-1) \times 0 \times (-1) - (-1) \times 1 \times 1 + 3 \times 1 \times 1 - 3 \times 4 \times (-1) = 32.$$

A careful arrangement of the six terms in Equation (3) gives

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}(\alpha_{23}\alpha_{31} - \alpha_{21}\alpha_{33}) + \alpha_{13}(\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31}).$$

Clearly,

$$\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = \alpha_1\beta_2 - \alpha_2\beta_1 \quad \text{and}$$

if  $A$  is a  $1 \times 1$  matrix  $[\alpha_1]$ , then  $\det A = \alpha_1$ .

When it comes to considering determinants of the fourth and higher orders, the calculation involved in writing out all the terms becomes prohibitive. So we study further properties of determinants and arrive at shorter methods of calculating  $\det A$ . However, certain special determinants can be evaluated directly from the definition, for example,



$$\begin{vmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{vmatrix} = \lambda_1 \lambda_2 \dots \lambda_n.$$

In particular,  $\det I_n = 1$  and  $\det \mathbf{0}_{n \times n} = 0$ .

### Problem Set 6.1

- Determine whether each of the following permutations is even or odd :  
 (a) (1, 3, 5, 6, 4, 2)      (b) (3, 4, 5, 2, 1, 6) ,  
 (c) (2, 6, 4, 3, 5, 1)      (d) (1, 4, 3, 2, 5) .
- Write down all possible permutations of the set  $A$  and separate the odd and even ones :  
 (a)  $A = \{1, 2, 3\}$       (b)  $A = \{1, 2, 3, 4\}$  .
- Find  $\det A$  for the given matrix  $A$  :

$$(a) \quad A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad (b) \quad A = \begin{bmatrix} 3 & 1 & 2 \\ 5 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} .$$

- Evaluate the following determinants :

$$(a) \quad \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \quad (b) \quad \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} \quad (c) \quad \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} .$$

- Prove that the solution of the system

$$\begin{aligned} \alpha_1 x_1 + \alpha_2 x_2 &= \delta_1 \\ \beta_1 x_1 + \beta_2 x_2 &= \delta_2 \end{aligned}$$

$$\text{is } x_1 = \frac{\begin{vmatrix} \delta_1 & \alpha_2 \\ \delta_2 & \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} \alpha_1 & \delta_1 \\ \beta_1 & \delta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}},$$

provided that  $\alpha_1 \beta_2 \neq \alpha_2 \beta_1$ .

- True or false ?

- If  $(a, b, c)$  is an even permutation, then  $(b, a, c)$  is odd.
- The sign attached to the product  $\alpha_{33}\alpha_{11}\alpha_{22}$  of a third order determinant is 'plus'.

$$(c) \begin{vmatrix} 0 & 1 & 3 \\ 2 & 0 & -4 \\ -5 & 5 & 0 \end{vmatrix} = 0.$$

$$(d) \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 0 \\ -5 & 5 & 5 \end{vmatrix} = 0.$$

$$(e) \text{ The equation } \begin{vmatrix} 2 - \lambda & 3 \\ 4 & 1 - \lambda \end{vmatrix} = 0 \text{ has no real root.}$$

(f) The determinant of a triangular matrix is the product of its diagonal entries.

$$(g) \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = -2.$$

$$(h) \begin{vmatrix} 1 & i \\ -i & -1 \end{vmatrix} = 0.$$

## 6.2 FUNDAMENTAL PROPERTIES OF DETERMINANTS

The fundamental properties of determinants, which follow immediately from the definition, are given in this article in the form of theorems. Of these, the proofs of Theorems 6.2.1 and 6.2.8 (see § 6.3) are rather complicated, and a beginner may skip them on his first reading. In this article  $A = (a_{ij})$  stands for an  $n \times n$  matrix.

**6.2.1 Theorem** *If the matrix  $B$  is obtained from  $A$  by an elementary row operation of type 1 (i.e. interchanging the rows), then  $\det B = -\det A$ .*

For the proof see § 6.3.

$$\text{Example 6.4} \quad \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 2 \\ 0 & 1 & 4 & 0 \\ 1 & -1 & 2 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 2 & 3 \\ 0 & 1 & 4 & 0 \\ -1 & 0 & 1 & 2 \end{vmatrix}$$

because the determinant on the right-hand side is just the determinant on the left-hand side with the second and fourth rows interchanged.

**6.2.2 Theorem** *If  $A$  has two identical rows, then  $\det A = 0$ .*

**Proof:**  $A$  can be considered as being obtained from itself by interchanging the two identical rows. So, by Theorem 6.2.1,  $\det A = -\det A$ . Hence,  $\det A = 0$ . ■

$$\text{Example 6.5} \quad \begin{vmatrix} 2 & 3 & -1 & 4 \\ 4 & 1 & 0 & -1 \\ 1 & 2 & -3 & 1 \\ 4 & 1 & 0 & -1 \end{vmatrix} = 0$$

without any further calculation.

**6.2.3 Theorem** If  $A$  has a row of zeros, then  $\det A = 0$ .

*Proof:*  $\det A = \sum_{(j_1, j_2, \dots, j_n)} \epsilon_{(j_1, j_2, \dots, j_n)} \alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{nj_n}$ . If the  $k$ -th row is the zero vector, then  $\alpha_{kj_k}$  is zero for every choice of  $j_k$ . Therefore, each product is zero and this gives  $\det A = 0$ . ■

$$\text{Example 6.6} \quad \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

without any further calculation.

**6.2.4 Theorem** If the  $k$ -th row vector  $r_k$  of  $A$  is the sum of two vectors  $b_k$  and  $c_k$  of  $V_n$ , then  $|A| = |B| + |C|$ , where  $B$  is the matrix obtained from  $A$  when  $r_k$  is replaced by  $b_k$ , and  $C$  is the matrix obtained from  $A$  when  $r_k$  is replaced by  $c_k$ .

Before we take up the proof, we shall illustrate the result by a simple example.

$$\begin{aligned} \text{Example 6.7} \quad & \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 & b_4 + c_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix} \\ &= \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ b_1 & b_2 & b_3 & b_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ c_1 & c_2 & c_3 & c_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{vmatrix}. \end{aligned}$$

*Proof of Theorem 6.2.4* Let  $b_k = (b_{k1}, b_{k2}, \dots, b_{kn})$  and  $c_k = (c_{k1}, c_{k2}, \dots, c_{kn})$ . Then  $\alpha_{kj_k} = b_{kj_k} + c_{kj_k}$  for every  $k$ . This shows that each product  $\alpha_{1j_1} \alpha_{2j_2} \dots \alpha_{kj_k} \dots \alpha_{nj_n}$  can be written as  $(\alpha_{1j_1} \alpha_{2j_2} \dots b_{kj_k} \dots \alpha_{nj_n}) + (\alpha_{1j_1} \alpha_{2j_2} \dots c_{kj_k} \dots \alpha_{nj_n})$ .

The algebraic sum of the first type of products with the appropriate signs gives  $\det B$ , and the algebraic sum of the second type of products with the appropriate signs gives  $\det C$ . Hence the result. ■

**6.2.5 Theorem** If the matrix  $B$  is obtained from  $A$  by an elementary operation of type II (i.e. multiplying any row vector by a nonzero scalar  $c$ ), then  $\det B = c \det A$ .

*Example 6.8* 
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ ca_2 & cb_2 & cc_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = c \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

*Proof of Theorem 6.2.5* Let the  $k$ -th row of  $A$  be multiplied by  $c$ . Then the  $k$ -th row of  $B$  is  $(ca_{k1}, ca_{k2}, \dots, ca_{kn})$ . So, in the expansion of  $\det B$  each product is of the form  $\alpha_{1j_1} \alpha_{2j_2} \dots (ca_{kj_k}) \dots \alpha_{nj_n}$ . This shows that every term of the expansion of  $\det B$  is a multiple of the corresponding term by a factor  $c$ , in the expansion of  $\det A$ . Hence,  $\det B = c \det A$ . ■

Note that Theorem 6.2.5 holds even if  $c = 0$ .

**6.2.6 Theorem** If the matrix  $B$  is obtained from  $A$  by an elementary row operation of type III (i.e. adding  $c$  times the  $s$ -th row to the  $k$ -th row), then  $\det B = \det A$ .

*Example 6.9* 
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 + \alpha c_1 & b_2 + \alpha c_2 & b_3 + \alpha c_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

*Example 6.10* 
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 & 6 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 5 - 1 & 6 - 2 & 7 - 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 4 & 4 \\ 4 & 4 & 4 \end{vmatrix} = 0,$$

by Theorem 6.2.2.

*Proof of Theorem 6.2.6* Let the  $k$ -th row of  $B$  be  $r_k + cr_s$ , where  $r_k$  and  $r_s$  are the  $k$ -th and  $s$ -th rows of  $A$ . So, by Theorem 6.2.4,

$$\det B = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{k1} & \alpha_{k2} & \dots & \alpha_{kn} \\ \vdots & \vdots & & \vdots \\ \alpha_{s1} & \alpha_{s2} & \dots & \alpha_{sn} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ c\alpha_{s1} & c\alpha_{s2} & \dots & c\alpha_{sn} \\ \vdots & \vdots & & \vdots \\ \alpha_{s1} & \alpha_{s2} & \dots & \alpha_{sn} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix} \begin{matrix} \rightarrow k\text{-th row.} \\ \\ \rightarrow s\text{-th row.} \end{matrix}$$

Applying Theorem 6.2.5, we obtain

$$\det B = \det A + c \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & \vdots & & \vdots \\ \alpha_{s1} & \alpha_{s2} & \cdots & \alpha_{sn} \\ \vdots & \vdots & & \vdots \\ \alpha_{s1} & \alpha_{s2} & \cdots & \alpha_{sn} \\ \vdots & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{vmatrix} \\ = \det A \quad (\text{by Theorem 6.2.2}). \quad \blacksquare$$

**6.2.7 Remark** Theorems 6.2.1, 6.2.5, and 6.2.6 tell us that the effect of a finite number of elementary row operations (of any or all of the three types) on  $\det A$  is to multiply it by a nonzero scalar.

**6.2.8 Theorem**  $\det A^T = \det A$ .

For the proof see § 6.3.

**Example 6.11**  $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 7 & 8 \\ 3 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 5 & 3 \\ 3 & 7 & -1 \\ 4 & 8 & 2 \end{vmatrix}.$

**6.2.9 Theorem** Theorems 6.2.1 to 6.2.6 are true if everywhere the word 'row(s)' is changed to 'column(s)'.

This is a consequence of Theorem 6.2.8.

**6.2.10 Remark** As a consequence of Theorem 6.2.9 the elementary operations can be performed as well on the columns of a determinant, with the same effect as that on rows.

**Example 6.12** Evaluate

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix} \\ |A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -6 \\ 0 & 2 & 4 \end{vmatrix}$$

by  $r_2 - 2r_1$  and  $r_3 + r_1$ . Using Theorem 6.2.8, we get

$$|A| = \begin{vmatrix} 1 & 0 & 0 \\ 2 & -5 & 2 \\ 3 & -6 & 4 \end{vmatrix} = 1 \times (-5) \times 4 - 1 \times 2 \times (-6) = -8$$

as all other products are zero.

**Problem Set 6.2**

1. If  $A$  is a square matrix of order  $n$  and  $\alpha$  a scalar, then prove that  $\det(\alpha A) = \alpha^n \det A$ .
2. Let  $A$  be a skew-symmetric matrix of odd order. Then prove that  $\det A = 0$ .
3. For a triangular matrix  $A$ , prove that  $\det A$  is the product of its diagonal entries.
4. Prove that

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \delta & \delta^2 \end{vmatrix} = (\alpha - \beta) \begin{vmatrix} 0 & 1 & \alpha + \beta \\ 1 & \beta & \beta^2 \\ 1 & \delta & \delta^2 \end{vmatrix} \\ = (\alpha - \beta)(\beta - \delta)(\delta - \alpha) \begin{vmatrix} 0 & 1 & \alpha + \beta \\ 0 & 0 & 1 \\ 1 & \delta & \delta^2 \end{vmatrix}.$$

5. Without expanding, prove that

$$(a) \begin{vmatrix} 1 & 1 & 3 \\ 2 & 9 & 1 \\ 4 & 11 & 7 \end{vmatrix} = 0 \quad (b) \begin{vmatrix} b+c & 1 & a \\ c+a & 1 & b \\ b+a & 1 & c \end{vmatrix} = 0$$

$$(c) \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

$$(d) \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$(e) \begin{vmatrix} a+b & b+c & c+a \\ p+q & q+r & r+p \\ x+y & y+z & z+x \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$(f) \begin{vmatrix} x & x+3 & x+6 \\ x+1 & x+4 & x+7 \\ x+2 & x+5 & x+8 \end{vmatrix} = 0 \quad (g) \begin{vmatrix} 4 & 7 & 10 \\ 10 & 13 & 16 \\ 20 & 23 & 26 \end{vmatrix} = 0$$

$$(h) \begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix} = 0 \quad (i) \begin{vmatrix} x-y & y-z & z-x \\ y-z & z-x & x-y \\ z-x & x-y & y-z \end{vmatrix} = 0.$$

6. True or false ?

(a) If every row of a determinant of order  $n$  ( $\neq 1$ ) is multiplied by  $k$  and every column by  $k'$ , then the value of the determinant is multiplied by  $kk'$ .

(b) If the diagonal entries of a determinant are multiplied by  $k$ , then the value of the determinant is multiplied by  $k$ .

$$(c) \begin{vmatrix} a & b & c \\ 2a & 2b & 2c \\ 3a & 3b & 3c \end{vmatrix} = 6abc. \quad (d) \begin{vmatrix} a & b & a+b \\ c & d & c+d \\ e & f & e+f \end{vmatrix} = 0.$$

$$(e) \begin{vmatrix} a_1 + a_2 & p_1 + p_2 & l_1 + l_2 \\ b_1 + b_2 & q_1 + q_2 & m_1 + m_2 \\ c_1 + c_2 & r_1 + r_2 & n_1 + n_2 \end{vmatrix} = \begin{vmatrix} a_1 & p_1 & l_1 \\ b_1 & q_1 & m_1 \\ c_1 & r_1 & n_1 \end{vmatrix} + \begin{vmatrix} a_2 & p_2 & l_2 \\ b_2 & q_2 & m_2 \\ c_2 & r_2 & n_2 \end{vmatrix}$$

$$(f) \begin{vmatrix} a & a' & a' \\ b & b' & b' \\ c & c' & c' \end{vmatrix} = - \begin{vmatrix} a & a' & a' \\ b & b' & b' \\ c & c' & c' \end{vmatrix}.$$

$$(g) \begin{vmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & d & e \end{vmatrix} = \begin{vmatrix} d & 0 & 0 \\ e & 0 & 0 \\ 0 & a & b \end{vmatrix}.$$

$$(h) \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ b_1 - c_1 & b_2 - c_2 & b_3 - c_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{vmatrix}.$$

### 6.3 PROOFS OF THEOREMS

To prove Theorem 6.2.1 we need the following two lemmas.

**6.3.1 Lemma** Let  $p$  be the number of inversions in a permutation  $P = (j_1, j_2, \dots, j_n)$  of the integers  $1, 2, 3, \dots, n$ . Let  $Q$  be the permutation obtained from  $P$  by changing two adjacent entries of  $P$ . Then the number of inversions in  $Q$  is  $p \pm 1$ .





Let  $A = (a_{ij})_{n \times n}$ . Then  $A^T = (a'_{ij})_{n \times n}$ , where  $a'_{ij} = a_{ji}$ . A typical term in  $\det A^T$  is

Taking product (1), without the sign, we have

Here  $(j_1, j_2, \dots, j_n)$  is a permutation of the set  $\{1, 2, \dots, n\}$ . Hence,

$$j_{k_2} = 2 \text{ for some } k_2 \in \{1, 2, \dots, n\}, \text{ where } k_2 \neq k_1,$$

$$j_{k_n} = n \text{ for some } k_n \in \{1, 2, \dots, n\}, \text{ where } k_n \neq k_{n-1}, \dots, k_2, k_1.$$

So product (2), without the sign factor, is

**i.c.**

Product (3) is the form of the product as it occurs in  $\det A$ . The sign attached to product (3) in  $\det A$  is  $\epsilon(k_1, k_2, \dots, k_n)$ . The sign attached to product (2) in  $\det A^T$  is, from product (1),  $\epsilon(j_1, j_2, \dots, j_n)$ .

If we can prove that these two signs are the same, we would be through. These signs depend on the number of inversions in the permutations  $P = (j_1, j_2, \dots, j_n)$  and  $Q = (k_1, k_2, \dots, k_n)$ . So we shall now analyse the process by which we obtained the rearrangement (3) from (2).

Both (2) and (3) represent the same product. The entries in product (2) have the column indices in the natural order  $1, 2, \dots, n$ , whereas the entries in product (3) have the row indices in the natural order. The order of rows in product (2) is  $P$  and the order of columns in product (3)

is  $Q$ . The passage from product (2) to product (3) is nothing but the rearrangement of  $P$  in the natural order. It is presented schematically in Table 6.1.

TABLE 6.1

Order of rows	Order of columns
$P = (j_1, j_2, \dots, j_n)$ $= (j_1, j_2, \dots, j_{k_1}, \dots, j_n)$	$(1, 2, \dots, n)$ $(1, 2, \dots, k_1, \dots, n)$
↓	↓
$(j_{k_1}, j_1, \dots, j_{k_1-1}, j_{k_1+1}, \dots, j_n)$	$(k_1, 1, 2, \dots, k_1-1, k_1+1, \dots, n)$
↓	↓
$(j_{k_1}, j_1, \dots, j_{k_2}, \dots, j_n)$	$(k_1, 1, 2, \dots, k_2, \dots, n)$
↓	↓
$(j_{k_1}, j_{k_2}, j_1, j_2, \dots, j_n)$	$(k_1, k_2, 1, 2, \dots, n)$
↓	↓
$(j_{k_1}, j_{k_2}, \dots, j_{k_n})$	$(k_1, k_2, \dots, k_n)$
↓	↓
$= (1, 2, \dots, n)$	$= Q$

The first line in Table 6.1 indicates the order of rows ( $--P$ ) and the order of columns (natural order) in product (2). The last line indicates the order of rows (natural order) and the order of columns ( $=Q$ ) in product (3).

The passage from product (2) to product (3) presented in Table 6.1 can be realised by successive interchanges of adjacent entries in the respective permutations. Since  $P = (j_1, j_2, \dots, j_n)$ , look for  $j_{k_1} = 1$  and bring it to the first position by interchanging it successively with all the entries preceding it. This needs a certain number of interchanges, say  $t$ . The effect of these interchanges is to reduce the number of inversions in  $P$  by  $t$ . If we make the *same kind and number* of interchanges in the order of columns, namely, the natural order  $1, 2, 3, \dots, n$ , we get  $(k_1, 1, 2, \dots, k_1-1, k_1+1, \dots, n)$ , the effect of which is to increase the number of inversions in  $(1, 2, \dots, n)$  by  $t$ .

Now we look for  $j_{k_2} = 2$  and bring it to the second position by making successive interchanges. Suppose this requires  $s$  interchanges. The *same*  $s$  interchanges carried out on the order of columns will bring  $k_2$

to the second position. The effect of this is to *reduce* the number of inversions in the order of rows by  $s$  and *increase* the number of inversions in the order of columns by  $s$ .

Carrying on this process, we arrive at  $(j_{k_1}, j_{k_2}, \dots, j_{k_n})$ , i.e. the natural order on the left and at  $Q = (k_1, k_2, \dots, k_n)$  on the right. Thus, the number of interchanges required to bring  $P$  to the natural order is the *same* as the number of interchanges required to bring  $(1, 2, \dots, n)$  to  $(k_1, k_2, \dots, k_n) = Q$ . Since the number of inversions in  $(1, 2, \dots, n)$  is zero, it follows that the number of inversions in  $P$  is the same as the number of inversions in  $Q$ . Hence,  $\epsilon(j_1, j_2, \dots, j_n) = \epsilon(k_1, k_2, \dots, k_n)$  and the proof is over. ■

## 6.4 COFACTORS

The expansion of an  $n$ -th order determinant is the sum of all the  $n!$  terms, each of which is a product obtained by taking one (and only one) element from each row and each column. Let  $A = (a_{ij})_{n \times n}$ . Suppose we collect all the terms in the expansion of  $\det A$ , in which the fixed element  $a_{ij}$  appears as a factor, and write down their sum in the factored form  $a_{ij}A_{ij}$ . Here  $A_{ij}$  denotes the factor remaining when the element  $a_{ij}$  is factored out. We call  $A_{ij}$  the *cofactor* of  $a_{ij}$  in  $\det A$ .

$$\text{Example 6.13} \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ + a_{12}(a_{23}a_{31} - a_{21}a_{33}) \\ + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

$$\begin{aligned} \text{So} \quad A_{11} &= \text{cofactor of } a_{11} = a_{22}a_{33} - a_{23}a_{32} \\ A_{12} &= \text{cofactor of } a_{12} = a_{23}a_{31} - a_{21}a_{33} \\ A_{13} &= \text{cofactor of } a_{13} = a_{21}a_{32} - a_{22}a_{31}. \end{aligned}$$

If we want any other cofactor, say  $A_{23}$ , we have to rearrange the above expansion and collect the terms containing  $a_{23}$ . Thus, we see that  $A_{23} = a_{12}a_{31} - a_{11}a_{32}$ .

An explicit formula for the cofactor of  $a_{ij}$  is given by Theorem 6.4.2. For this we need the following definition.

**6.4.1 Definition** The determinant obtained by deleting the  $i$ -th row and the  $j$ -th column in  $\det A$  is denoted by  $\Delta_{ij}$ .

$\Delta_{ij}$  is also called the *minor* of the element  $a_{ij}$  in  $\det A$ , but we shall not use this terminology.

$$\text{Example 6.14} \quad \text{For the determinant } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\Delta_{11} = \begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix}, \quad \Delta_{21} = \begin{vmatrix} \alpha_{13} & \alpha_{12} \\ \alpha_{33} & \alpha_{32} \end{vmatrix}$$

and so on.

**Example 6.15** If  $A = (\alpha_{ij})_{n \times n}$ , then

$$\Delta_{nn} = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1, n-1} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2, n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1, 1} & \alpha_{n-1, 2} & \cdots & \alpha_{n-1, n-1} \end{vmatrix}.$$

**6.4.2 Theorem** Let  $A = (\alpha_{ij})_{n \times n}$ . Then for each  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, n$ ,

$$A_{ij} = (-1)^{i+j} \Delta_{ij}. \quad (1)$$

*Proof:* First, we shall prove a special case of the theorem, namely, that  $A_{nn} = \Delta_{nn}$ . To prove this we collect all the terms involving  $\alpha_{nn}$ . They are of the form

$$\epsilon(j_1, j_2, \dots, j_{n-1}, n) \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{n-1, j_{n-1}} \alpha_{nn},$$

where  $(j_1, j_2, \dots, j_{n-1})$  is a permutation of the set  $\{1, 2, \dots, (n-1)\}$ . The sum of all these terms is  $\sigma_{nn} A_{nn}$ . But

$$\epsilon(j_1, j_2, \dots, j_{n-1}, n) = \epsilon(j_1, j_2, \dots, j_{n-1}),$$

since the integer  $n$  occurring last does not increase or decrease the number of inversions in  $(j_1, j_2, \dots, j_{n-1})$ . Therefore,

$$A_{nn} = \sum \epsilon(j_1, j_2, \dots, j_{n-1}) \alpha_{1j_1} \alpha_{2j_2} \cdots \alpha_{n-1, j_{n-1}}$$

$$= \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1, n-1} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2, n-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{n-1, 1} & \alpha_{n-1, 2} & \cdots & \alpha_{n-1, n-1} \end{vmatrix} \quad \text{(by definition of a determinant)}$$

$$= \Delta_{nn}.$$

Now we take up the general case and reduce it to the special case. We have

$$\det A = \begin{vmatrix} P & Q & R \\ \cdots & \cdots & \cdots \\ S & \alpha_{ij} & T \\ \cdots & \cdots & \cdots \\ U & V & W \end{vmatrix},$$

where  $P, Q, R, S, T, U, V$ , and  $W$  are mere abbreviations for the remaining entries.  $U, V$ , and  $W$  have  $(n-i)$  rows;  $R, T$ , and  $W$  have  $(n-j)$

columns;  $S$  and  $T$  have only one row, and  $Q$ ,  $V$  have only one column. Clearly,

$$\Delta_{ij} = \begin{vmatrix} P & R \\ U & W \end{vmatrix}. \quad (2)$$

Shifting the  $i$ -th row by successive interchanges with the  $(n - i)$  rows of  $U$ ,  $V$ , and  $W$ , we get

$$\det A = (-1)^{n-i} \begin{vmatrix} P & Q & R \\ U & V & W \\ S & \alpha_{ij} & T \end{vmatrix} \quad (\text{by Theorem 6.2.1}).$$

Again, by another series of interchanges of the  $j$ -th column with  $(n - j)$  columns of  $R$ ,  $W$ , and  $T$  and by applying Theorems 6.2.1 and 6.2.9, we get

$$\det A = (-1)^{n-i} (-1)^n \begin{vmatrix} P & R & Q \\ U & W & V \\ S & T & \alpha_{ij} \end{vmatrix}.$$

So the cofactor of  $\alpha_{ij}$  in  $\det A$

$$\begin{aligned} &= (-1)^{i+j} \times \text{cofactor of } \alpha_{ij} \text{ in } \begin{vmatrix} P & R & Q \\ U & W & V \\ S & T & \alpha_{ij} \end{vmatrix} \\ &= (-1)^{i+j} \times \begin{vmatrix} P & R \\ U & W \end{vmatrix}, \end{aligned}$$

by the preceding part of the proof. Hence,  $A_{ij} = (-1)^{i+j} \Delta_{ij}$  by Equation (2). ■

**Example 6.16** Consider

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The cofactor of  $h$  occurring in the second row, first column, is

$$(-1)^{2+1} \begin{vmatrix} h & g \\ f & c \end{vmatrix} = -(hc - gf) = fg - ch.$$

The cofactor of  $f$  occurring in the third row, second column, is

$$(-1)^{3+2} \begin{vmatrix} a & g \\ h & f \end{vmatrix} = gh - af.$$

Since this determinant is symmetric, the cofactors of elements, which are symmetrically placed with respect to the diagonal, are the same. So, for example, we need not distinguish between  $\alpha_{12} = h$  and  $\alpha_{21} = h$ . Both have the same cofactor. (Check!) Denoting the cofactors by capital letters, we have

$$\begin{aligned} A &= bc - f^2, & B &= ca - g^2, & C &= ab - h^2, \\ F &= gh - af, & G &= hf - bg, & H &= fg - ch. \end{aligned}$$

The  $n!$  terms that occur in the expansion of  $\det A$  can be separated into  $n$  groups as follows :

- (1) all terms containing  $\alpha_{11}$
- (2) all terms containing  $\alpha_{12}$
- $\vdots$
- ( $n$ ) all terms containing  $\alpha_{1n}$ .

Each of these groups contains  $(n-1)!$  terms, since this is the number of terms in any  $\Delta_{1j}$ , which is a determinant of order  $(n-1)$ . These groups are mutually exclusive, because when a term contains  $\alpha_{1j}$ , it cannot contain any other element from the first row. Thus, there are  $n$  mutually exclusive groups, each containing  $(n-1)!$  terms. Therefore, the total number of terms in all these groups is  $n \times (n-1)! = n!$ . Hence, it follows that all the terms in the expansion of  $\det A$  are exhausted by this grouping. Thus, we have

$$\det A = \alpha_{11}A_{11} + \alpha_{12}A_{12} + \dots + \alpha_{1n}A_{1n}. \quad (3)$$

If, instead of the first row, we focus our attention on the elements of the  $i$ -th row and collect terms involving  $\alpha_{i1}, \alpha_{i2}, \dots$ , etc., we get, by the same argument as that which resulted in Equation (3) :

$$\det A = \alpha_{i1}A_{i1} + \alpha_{i2}A_{i2} + \dots + \alpha_{in}A_{in}. \quad (4)$$

Similarly, we can work with the elements of any column, say the  $j$ -th column, to get

$$\det A = \alpha_{1j}A_{1j} + \alpha_{2j}A_{2j} + \dots + \alpha_{nj}A_{nj}. \quad (5)$$

Thus, we have proved the following theorem.

**6.4.3 Theorem** *If  $A = (\alpha_{ij})$  is a square matrix of order  $n$  and  $A_{ij}$  denotes the cofactor of  $\alpha_{ij}$  in  $\det A$ , then, for each  $i = 1, 2, \dots, n$*

$$\det A = \sum_{k=1}^n \alpha_{ik}A_{ik} \quad (6)$$

*and, for each  $j = 1, 2, \dots, n$ ,*

$$\det A = \sum_{k=1}^n \alpha_{kj}A_{kj}. \quad (7)$$

Equation (6) is called the *expansion of  $\det A$  in terms of the  $i$ -th row* and Equation (7) is called the *expansion of  $\det A$  in terms of the  $j$ -th column*.

**Example 6.17** Evaluate

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Expansion in terms of the *first row* gives

$$\Delta = aA + hH + gG.$$

Expansion in terms of the *second row* gives

$$\Delta = hH + bB + fF,$$

and expansion in terms of the *third column* gives

$$\Delta = gG + fF + cC.$$

Taking any one of these, we get

$$\begin{aligned} \Delta &= aA + hH + gG \\ &= a(bc - f^2) + h(fg - ch) + g(fh - bg) \\ &= abc + 2fgh - af^2 - bg^2 - ch^2. \end{aligned}$$

In order to evaluate a determinant, we usually use the theorems of § 6.2 to introduce as many zeros as possible in any row (column) and then expand in terms of that row (column).

**Example 6.18** Evaluate

$$\Delta = \begin{vmatrix} 2 & 1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ 3 & -2 & 1 & 0 \\ -1 & 4 & 2 & 1 \end{vmatrix}.$$

Multiplying the first row by 2 and adding it to the third row, we get

$$\Delta = \begin{vmatrix} 2 & 1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ 7 & 0 & -5 & 8 \\ -1 & 4 & 2 & 1 \end{vmatrix} \underset{r_3 - 4r_1}{=} \begin{vmatrix} 2 & 1 & -3 & 4 \\ 1 & 0 & -1 & 2 \\ 7 & 0 & -5 & 8 \\ -9 & 0 & 14 & -15 \end{vmatrix}.$$

Expanding  $\Delta$  in terms of the second column, we get

$$\Delta = \begin{vmatrix} 1 & -1 & 2 \\ 7 & -5 & 8 \\ -9 & 14 & -15 \end{vmatrix} \underset{\substack{c_2 + c_1, \\ c_3 - 2c_1}}{=} - \begin{vmatrix} 1 & 0 & 0 \\ 7 & 2 & -6 \\ -9 & 5 & 3 \end{vmatrix},$$

where  $c_i$  denotes the  $i$ -th column. Now expansion in terms of the first row gives

$$\Delta = - \begin{vmatrix} 2 & -6 \\ 5 & 3 \end{vmatrix} = -(6 + 30) = -36.$$

**6.4.4 Theorem** Let the row vectors of the square matrix  $A = (a_{ij})_{n \times n}$  be  $r_1, r_2, \dots, r_n$ . Let the vector  $R_i$  denote the vector of cofactors, in  $\det A$ , of the element of  $r_i$ . In other words,

$$r_i = (a_{i1}, a_{i2}, \dots, a_{ij}, \dots, a_{in})$$

and  $R_i = (A_{i1}, A_{i2}, \dots, A_{ij}, \dots, A_{in})$ .

Then  $r_i \cdot R_i = \det A$  for all  $i = 1, 2, \dots, n$ ,

and  $r_i \cdot R_j = 0$  if  $i \neq j$ .

In short,  $r_i \cdot R_j = \delta_{ij} \det A$  for all  $i, j = 1, 2, \dots, n$ .

*Proof:* The first result is only a restatement of Equation (6). To prove the second result, let us construct a matrix  $B$  by replacing the  $j$ -th row of  $A$  by the  $i$ -th row of  $A$ . Note that the  $i$ -th row of  $A$  remains as it is in this construction. So

$$\det B = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \rightarrow i\text{-th row} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \dots & a_{jn} \rightarrow j\text{-th row} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

This determinant is zero, because two rows are identical. On the other hand, we can expand  $\det B$  in terms of its  $j$ -th row. Since the deletion of the  $j$ -th row and the  $k$ -th column of  $\det B$  gives the same  $\Delta_{jk}$  as the deletion of the  $j$ -th row and the  $k$ -th column of  $\det A$ , we get

$$\begin{aligned} 0 &= \det B = a_{i1}A_{j1} + a_{i2}A_{j2} + \dots + a_{in}A_{jn} \\ &= r_i \cdot R_j. \blacksquare \end{aligned}$$

**6.4.5 Remark** A similar result as in Theorem 6.4.4 is true for columns.

**6.4.6 Corollary** Let  $A$  be the matrix  $(a_{ij})_{n \times n}$  and  $B$  the matrix  $(\beta_{ij})_{n \times n}$ , where  $\beta_{ij} = A_{ji}$ . Then

$$AB = BA = \begin{bmatrix} \Delta & 0 & 0 & \dots & 0 \\ 0 & \Delta & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \Delta \end{bmatrix} = \Delta I_n,$$

where  $\Delta = \det A$ . Consequently,  $\det(AB) = \Delta^n = \det(BA)$ .

The proof is obvious when we carry out the multiplication of  $A$  and  $B$  and repeatedly use Theorem 6.4.4.



**6.4.7 Definition** The matrix  $B$  of Corollary 6.4.6 is called the *adjoint of*  $A$  and is written as  $\text{adj } A$ .

**Example 6.19** Find the adjoint of

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}.$$

The cofactors of the different entries in  $\det A$  are

$$\begin{aligned} A_{11} &= -1, & A_{12} &= 1, & A_{13} &= -1, \\ A_{21} &= 1, & A_{22} &= 1, & A_{23} &= -3, \\ A_{31} &= -1, & A_{32} &= -1, & A_{33} &= 1. \end{aligned}$$

Therefore,

$$\text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -3 & 1 \end{bmatrix}.$$

The adjoint is of significance in finding the inverse of a nonsingular matrix, as will be shown at the end of § 6.5.

### Problem Set 6.4

1. In Corollary 6.4.6 we repeatedly use Theorem 6.4.4. How many times do we use it?
2. Prove Remark 6.4.5.
3. Evaluate the following determinants:

$$\begin{aligned} \text{(a)} \quad & \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 3 & 2 \end{vmatrix} & \text{(b)} \quad & \begin{vmatrix} 1 & 2 & 4 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{vmatrix} & \text{(c)} \quad & \begin{vmatrix} 1 & 3 & 4 \\ 1 & 6 & 10 \\ 1 & 10 & 20 \end{vmatrix} \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & \begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix} & \text{(e)} \quad & \begin{vmatrix} 3 & 8 & 7 & 6 \\ 7 & 4 & 10 & 2 \\ 6 & 8 & 5 & 8 \\ 9 & 5 & 3 & 9 \end{vmatrix} & \text{(f)} \quad & \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}. \end{aligned}$$

4. Without expanding, prove that

$$\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}.$$

5. Prove that

$$(a) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x - y)(y - z)(z - x)$$

$$(b) \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = (x - y)(y - z)(z - x)$$

$$(c) \begin{vmatrix} x + a & b & c & d \\ a & x + b & c & d \\ a & b & x + c & d \\ a & b & c & x + d \end{vmatrix} = x^3(x + a + b + c + d).$$

6. If  $\omega_1, \omega_2$ , and  $\omega_3$  are the three cube roots of unity, then prove that

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{vmatrix} = \prod_{i=1}^3 (x_1 + x_2\omega_i + x_3\omega_i^2)$$

This determinant is called a *circulant* of the third order. Write down a circulant of order  $n$ . Write also its value.

7. (a) Prove that the equation of a circle through three points  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$  is given by

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

(b) Determine the equation of the sphere passing through the points  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ ,  $(x_3, y_3, z_3)$ , and  $(x_4, y_4, z_4)$  in the determinant form.

8. Solve the equation

$$\begin{vmatrix} x + a & b & c \\ c & x + b & a \\ a & b & x + c \end{vmatrix} = 0.$$

9. Without expanding, prove that

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

10. Calculate the adjoint of matrices of Problem 2, Problem Set 5.5.

11. True or false?

- (a) In the matrix  $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$  the cofactor  $A_{21}$  is  $-\sin \theta$ .
- (b) There exist determinants none of whose entries is zero, but for which every cofactor is zero.
- (c) If a matrix is nonzero, then its adjoint is also nonzero.
- (d)  $\text{adj}(\text{adj } A) = A$ .
- (e)  $\text{adj}(\alpha I_n) = \alpha^{n-1} I_n$ .
- (f) If  $\text{adj } A$  is a diagonal matrix, then  $A$  is also diagonal.

## 6.5 DETERMINANT MINORS AND RANK OF A MATRIX

Given a matrix  $A$ , we can get many smaller matrices from it by simply deleting a certain number of its rows and/or columns. Matrices thus obtained are called *submatrices* of  $A$ .

**6.5.1 Definition** The determinant of a square submatrix  $B$  of  $A$  is called a *determinant minor* of  $A$  or simply a *minor* of  $A$ .

**Example 6.20** Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}.$$

Then  $B = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 10 & 11 \end{bmatrix}$  is a submatrix of  $A$  and  $\begin{vmatrix} 1 & 3 \\ 9 & 11 \end{vmatrix}$  is a minor of  $A$  and also of  $B$ .

**6.5.2 Lemma** (a) Let  $A$  be an  $m \times n$  matrix. Apply some elementary row operations to obtain a new matrix  $A_1$ . Delete some columns of  $A_1$  to get a submatrix  $B_1$  of  $A_1$ . The same submatrix  $B_1$  could also have been obtained if we had first deleted the same columns from  $A$  itself to obtain a submatrix  $B$  and then applied the same elementary row operations to  $B$ .

(b) If  $B_1$  is a square matrix (and therefore  $B$  also), then  $\det B$  is equal to some nonzero scalar multiple of  $\det B_1$ .

The proof is obvious, because of the way elementary row operations work. Deletion of a column or columns will not affect the result of the elementary row operations on the remaining submatrix.

Part (b) is only a restatement of Remark 6.2.7.

**6.5.3 Theorem** If  $A$  is an  $m \times n$  matrix, then the rows of  $A$  are LI iff  $A$  has a nonzero minor of order  $m$ .

*Proof:* Let  $A_1$  be the row-reduced echelon form of  $A$ . Suppose the rows of  $A$  are LI. Then  $A_1$  would contain  $m$  nonzero rows. The  $m$  columns containing the 1's at the steps will also be nonzero. Delete the remaining columns of  $A_1$  and obtain an  $m \times m$  submatrix  $B_1$  of  $A_1$ . Obviously,  $B_1 = I_m$  and  $\det B_1 = \det I_m = 1$ . By Lemma 6.5.2 (b), the corresponding square submatrix  $B$  of the matrix  $A$  will have the property that  $\det B$  is a nonzero scalar multiple of  $\det B_1$ , which is 1. So  $\det B \neq 0$ . Thus,  $A$  has a nonzero minor of order  $m$ .

Conversely, suppose  $A$  has a nonzero minor of order  $m$ . Let  $B$  be the corresponding submatrix. We shall now prove that  $A_1$  contains  $m$  nonzero rows. Suppose, if possible,  $A_1$  has a zero row. The same operations that reduced  $A$  to  $A_1$  will reduce the submatrix  $B$  to  $B_1$ . Since  $B_1$  is of order  $m \times m$  and at least one of the  $m$  rows of  $A_1$  is zero, it follows that one of the rows of  $B_1$  is zero. So  $\det B_1 = 0$ . But, by Lemma 6.5.2 (b),  $\det B$  is equal to some nonzero scalar multiple of  $\det B_1$ , which is zero. This is a contradiction. Thus, all the rows of  $A_1$  are nonzero. In other words, the rows of  $A$  are LI. ■

**6.5.4 Theorem** The rank  $r$  of an  $m \times n$  matrix  $A$  is the order of the largest nonzero minor of  $A$ .

In other words, the matrix  $A$  is of rank  $r$  iff there exists a minor of order  $r$  which is nonzero, and every minor of order  $s > r$  is zero.

*Proof:* Let  $p$  be the integer such that there exists a minor of order  $p$ , which is nonzero, and every larger minor is zero. We shall now prove that  $r = p$ .

Since  $A$  has rank  $r$ ,  $r$  rows of  $A$  are LI. Choose such  $r$  rows. Let  $B$  be the submatrix of  $A$  obtained by deleting the remaining rows.  $B$  is an  $r \times n$  matrix, all of whose rows are LI. Therefore, by Theorem 6.5.3,  $B$  contains a nonzero minor of order  $r$ . Hence,  $r \leq p$ .

On the other hand, look at the nonzero minor of order  $p$ . This gives rise to a  $p \times p$  submatrix  $B$ . This in turn gives rise to a  $p \times n$  submatrix  $C$  of  $A$ .  $C$  contains a  $p \times p$  nonzero minor. Hence, by Theorem 6.5.3, the rows of  $C$  are LI. This means that  $A$  contains  $p$  LI rows and so  $p \leq r$ . Thus,  $r = p$ . ■

**6.5.5 Corollary** A square matrix  $A$  of order  $n$  is nonsingular iff  $\det A \neq 0$ .

The proof is left to the reader.

**Example 6.21** Find the rank of  $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 6 & -1 & 1 \end{bmatrix}$  by examining the determinants minors.

$$|A| = \begin{vmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 6 & -1 & 1 \end{vmatrix} \xrightarrow{r_3 - 2r_1} \begin{vmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{vmatrix} = 0.$$

So the rank of  $A$  is less than 3. The minor  $\begin{vmatrix} 1 & -3 \\ -1 & 1 \end{vmatrix} \neq 0$ .

So the rank of  $A$  is 2.

### INVERSE OF A MATRIX

We are now ready to give another method of finding the inverse of a nonsingular matrix :

Let  $A = (a_{ij})_{n \times n}$  be a nonsingular matrix. By Corollary 6.4.6, we have

$$AB = BA = \Delta I,$$

where  $B = \text{adj } A$  and  $\Delta = \det A$ . Since  $\Delta \neq 0$ , we get

$$A \begin{pmatrix} 1 \\ \Delta B \end{pmatrix} = \begin{pmatrix} 1 \\ \Delta B \end{pmatrix} A = I.$$

Therefore,  $A^{-1} = \frac{1}{\Delta} B = \frac{1}{\Delta} \text{adj } A$

$$= \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}.$$

**Example 6.22** Find the inverse of the matrix of Example 6.19.

We have

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -3 & 1 \end{bmatrix}.$$

Here  $\det A = -2$ . Therefore,

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & -1/2 \end{bmatrix}.$$

### Problem Set 6.5

1. Give explicitly the proofs of the 'if' part and the 'only if' part of Theorem 6.5.4.
2. Use determinant minors to calculate the rank of the matrices of Problem 1, Problem Set 5.5.
3. Using determinants, prove that the matrices of Problem 2, Problem Set 5.5, are nonsingular.
4. Use Theorem 6.5.3 to solve Problems 1 and 2, Problem Set 3.5.
5. Calculate the inverse by adjoint method for nonsingular matrices of Problem 2, Problem Set 5.5.
6. True or false ?
  - (a) If  $k$ -th order minors of a matrix  $A$  are all zero, then the rank of  $A$  is  $k - 1$ .
  - (b) If the rank of  $A$  is  $k$ , then every  $k$ -th order minor is nonzero.

(c) In the matrix  $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$  the minor  $\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$  is zero.

Therefore, the rank of  $A$  is 1.

- (d) If a row and a column of an  $n \times n$  matrix  $A$  are zero, then the rank of  $A$  is less than or equal to  $n - 2$ .

## 6.6 PRODUCT OF DETERMINANTS

In this article we shall study the determinant of the product of two square matrices. In this connection we have the following theorem.

**6.6.1 Theorem** *If  $A$  and  $B$  are two square matrices of the same order, then*  

$$\det(AB) = (\det A)(\det B).$$

The proof of this theorem is rather complicated. So we shall be satisfied with verifying the theorem for  $2 \times 2$  matrices.

$$\text{Let } A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}.$$

Then  $\det A = \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$   
 and  $\det B = \beta_{11}\beta_{22} - \beta_{12}\beta_{21}$ .

We shall now calculate  $\det(AB)$ . Since

$$AB = \begin{bmatrix} \alpha_{11}\beta_{11} + \alpha_{12}\beta_{21} & \alpha_{11}\beta_{12} + \alpha_{12}\beta_{22} \\ \alpha_{21}\beta_{11} + \alpha_{22}\beta_{21} & \alpha_{21}\beta_{12} + \alpha_{22}\beta_{22} \end{bmatrix},$$

we have  $\det(AB) = (\alpha_{11}\beta_{11} + \alpha_{12}\beta_{21})(\alpha_{21}\beta_{12} + \alpha_{22}\beta_{22})$   
 $- (\alpha_{11}\beta_{12} + \alpha_{12}\beta_{22})(\alpha_{21}\beta_{11} + \alpha_{22}\beta_{21})$ .

A merciless simplification will show that  $\det(AB) = (\det A)(\det B)$ .

Though we have not proved Theorem 6.6.1, we shall be using it in the sequel. A complete proof can be found in advanced texts on the subject.

**6.6.2 Corollary** *If  $A$  is a nonsingular matrix, then*

$$\det(A^{-1}) = (\det A)^{-1} = \frac{1}{\det A}.$$

*Proof:* We have  $AA^{-1} = I$ .

So  $\det(AA^{-1}) = \det I = 1$   
 or  $(\det A)(\det A^{-1}) = 1$ .

Therefore,  $\det A \neq 0$  and the result follows. ■

Theorem 6.6.1 can be rewritten in the following form.

**6.6.3 Theorem (Reworded)** *If  $A = (\alpha_{ij})$  and  $B = (\beta_{ij})$  are square matrices of order  $n$ , then  $(\det A)(\det B) = \det C$ , where  $C = (c_{ij})_{n \times n}$  and*

$$c_{ij} = \sum_{k=1}^n \alpha_{ik}\beta_{kj}.$$

**Example 6.23** Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 6 & 1 \\ 1 & 5 & 3 \end{bmatrix}.$$

$$\begin{aligned} \text{Then } \det A &= \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 3 & 6 \\ 0 & 2 & 2 \end{vmatrix} \\ &\quad \begin{matrix} r_2 - 2r_1, \\ r_3 - r_1 \end{matrix} \\ &= \begin{vmatrix} 3 & 6 \\ 2 & 2 \end{vmatrix} = -6, \end{aligned}$$

$$\begin{aligned} \text{and } \det B &= \begin{vmatrix} -1 & 2 & 0 \\ 3 & 6 & 1 \\ 1 & 5 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 3 & 0 \\ 0 & 12 & 1 \\ 0 & 7 & 3 \end{vmatrix} = (-1) \begin{vmatrix} 12 & 1 \\ 7 & 3 \end{vmatrix} = -29. \end{aligned}$$

$$\text{Also } AB = \begin{bmatrix} -2 & -3 & -3 \\ 11 & 42 & 15 \\ 6 & 19 & 5 \end{bmatrix} \text{ and}$$

$$\begin{aligned} \det(AB) &= \begin{vmatrix} -2 & -3 & -3 \\ 11 & 42 & 15 \\ 6 & 19 & 5 \end{vmatrix} \xrightarrow{r_3 + 3r_1} \begin{vmatrix} -2 & -3 & -3 \\ 11 & 42 & 15 \\ 0 & 10 & -4 \end{vmatrix} \xrightarrow{r_2 + 5r_1} \begin{vmatrix} -2 & -3 & -3 \\ 1 & 27 & 0 \\ 0 & 10 & -4 \end{vmatrix} \\ &\xrightarrow{r_1 + 2r_2} \begin{vmatrix} 0 & 51 & -3 \\ 1 & 27 & 0 \\ 0 & 10 & -4 \end{vmatrix} = - \begin{vmatrix} 51 & -3 \\ 10 & -4 \end{vmatrix} = -(-204 + 30) = 174 \\ &= (\det A)(\det B). \end{aligned}$$

### Problem Set 6.6

1. In the proof of Corollary 6.6.2 justify the step 'Therefore  $\det A \neq 0$ '.
2. If  $A$  and  $B$  are square matrices of order  $n$ , then prove that  $\det(A^T B) = \det(AB^T) = \det(A^T B^T) = \det(AB)$ .
3. If  $A$  is a square matrix, then prove that  $\det(A^n) = (\det A)^n$  for all positive integers  $n$ . (Hint : Use induction.)
4. Prove that the determinant of an idempotent matrix is either 0 or 1.
5. Evaluate  $\det A$ , if  $A$  is a nilpotent matrix.
6. Prove that

$$(a) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 3 & a+b+c & a^2+b^2+c^2 \\ a+b+c & a^2+b^2+c^2 & a^3+b^3+c^3 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^4+b^4+c^4 \end{vmatrix}$$

$$(b) \begin{vmatrix} 0 & z & y \\ z & 0 & x \\ y & x & 0 \end{vmatrix} = \begin{vmatrix} y^2+z^2 & xy & xz \\ xy & z^2+x^2 & yz \\ zx & yz & x^2+y^2 \end{vmatrix}$$

$$(c) \begin{vmatrix} 2yz - x^2 & z^2 & y^2 \\ z^2 & 2zx - y^2 & x^2 \\ y^2 & x^2 & 2xy - z^2 \end{vmatrix} = \begin{vmatrix} x & y & z \\ y & z & x \\ z & x & y \end{vmatrix}^2.$$



7. Prove that

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix},$$

where  $A_{ij}$  is the cofactor of  $\alpha_{ij}$ .

8. True or false ?

- If  $A$  is involutory, then  $\det A = 1$ .
- If  $(\det A)(\det B) = 1$ , then  $AB = I$ .
- $\det(A + B) = \det A + \det B$ .
- $(\det(cA))(\det(B)) = c(\det A)(\det B)$ , where  $c$  is a scalar.
- $\det(AB) = \det(BA)$ .
- $\det \in \mathcal{F}R(\mathbb{M}_n, n)$ .
- If  $A$  and  $B$  are two square matrices such that  $A = C^{-1}BC$  for some invertible matrix  $C$ , then  $\det A = \det B$ .

## 6.7 APPLICATIONS TO LINEAR EQUATIONS

Recall Theorem 5.8.2. The system of  $n$  equations in  $n$  unknowns

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1n}x_n &= \beta_1 \\ \alpha_{21}x_1 + \alpha_{22}x_2 + \dots + \alpha_{2n}x_n &= \beta_2 \\ \vdots & \vdots \\ \alpha_{n1}x_1 + \alpha_{n2}x_2 + \dots + \alpha_{nn}x_n &= \beta_n \end{aligned} \quad (1)$$

has a unique solution iff the matrix  $A = (\alpha_{ij})_{n \times n}$  is nonsingular, i.e.  $\det A \neq 0$  (cf Corollary 6.5.5). We shall now give a method, using determinants, of finding the solution when  $\det A \neq 0$ .

Let the cofactor of  $\alpha_{ij}$  in  $\det A$  be  $A_{ij}$ . Multiply the first equation of system (1) by  $A_{11}$ , the second by  $A_{21}$ , and so on, the last equation by  $A_{n1}$ , and add. We get

$$\begin{aligned} & x_1(\alpha_{11}A_{11} + \alpha_{21}A_{21} + \dots + \alpha_{n1}A_{n1}) \\ & + x_2(\alpha_{12}A_{11} + \alpha_{22}A_{21} + \dots + \alpha_{n2}A_{n1}) \\ & + \dots \\ & + x_n(\alpha_{1n}A_{11} + \alpha_{2n}A_{21} + \dots + \alpha_{nn}A_{n1}) \\ & = \beta_1A_{11} + \beta_2A_{21} + \dots + \beta_nA_{n1}. \end{aligned}$$

In view of Theorem 6.4.4 and Remark 6.4.5, this equation becomes

$$x_1 \det A = \beta_1A_{11} + \beta_2A_{21} + \dots + \beta_nA_{n1}$$

$$= \begin{vmatrix} \beta_1 & \alpha_{12} & \dots & \alpha_{1n} \\ \beta_2 & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots \\ \beta_n & \alpha_{n2} & \dots & \alpha_{nn} \end{vmatrix}.$$

The last equality can be seen by expanding this new determinant in terms of its first column :

$$\text{So } x_1 = \frac{1}{\det A} \begin{vmatrix} \beta_1 & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ \beta_2 & \alpha_{22} & \alpha_{23} & \dots & \alpha_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ \beta_n & \alpha_{n2} & \alpha_{n3} & \dots & \alpha_{nn} \end{vmatrix}.$$

In the same manner, if we multiply the first equation by  $A_{1j}$ , the second by  $A_{2j}$ , and so on, the last by  $A_{nj}$ , and add, we get

$$x_j = \frac{1}{\det A} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1,j-1} & \beta_1 & \alpha_{1,j+1} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2,j-1} & \beta_2 & \alpha_{2,j+1} & \dots & \alpha_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{n,j-1} & \beta_n & \alpha_{n,j+1} & \dots & \alpha_{nn} \end{vmatrix} \quad (2)$$

for all  $j = 1, 2, \dots, n$ .

This method of solving a system of  $n$  equations in  $n$  unknowns is known as *Cramer's rule*. This can also be derived as follows :

System (1) can be written in matrix notation as

$$Ax = b, \quad (3)$$

where  $A$  is the square matrix  $(\alpha_{ij})_{n \times n}$ ,  $x = (x_1, x_2, \dots, x_n)^T$ ,  $b = (\beta_1, \beta_2, \dots, \beta_n)^T$ . Since  $A$  is nonsingular, we get

$$A^{-1}(Ax) = A^{-1}b$$

or

$$x = A^{-1}b. \quad (4)$$

But

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A \quad (\text{cf } \S 6.4).$$

Hence,

$$x = \frac{1}{\det A} (\operatorname{adj} A)b \quad (5)$$

$$\text{or } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1j} & A_{2j} & \dots & A_{nj} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

Therefore,

$$x_j = \frac{1}{\det A} (\beta_1 A_{1j} + \beta_2 A_{2j} + \dots + \beta_n A_{nj})$$

$$= \frac{1}{\det A} \begin{vmatrix} \alpha_{11} & \alpha_{12} & \cdots & \beta_1 & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \beta_2 & \cdots & \alpha_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \beta_n & \cdots & \alpha_{nn} \end{vmatrix}.$$

$\uparrow$   
 $j$ -th column

**Example 6.24** Solve, by Cramer's rule,

$$\begin{aligned} 2x - 3y + z &= 1 \\ x + y - z &= 0 \\ x - 2y + z &= -1. \end{aligned}$$

Here

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$$

and  $\det A = 1$ . Therefore,

$$x = \frac{1}{1} \begin{vmatrix} 1 & -3 & 1 \\ 0 & 1 & -1 \\ -1 & -2 & 1 \end{vmatrix} = 1, \quad y = \frac{1}{1} \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -1,$$

$$\text{and} \quad z = \frac{1}{1} \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -4.$$

Note that if  $\det A = 0$ , then  $A$  has a rank less than  $n$  and so  $Ax = 0$  has nontrivial solutions. In this case  $Ax = b$  may or may not have a solution. Even if it has a solution the solution will not be unique. Conversely, if  $Ax = 0$  has nontrivial solutions, then  $\text{rank } A$  is less than  $n$ . So  $\det A = 0$ . Thus, we have proved the following theorem.

**6.7.1 Theorem** If  $A = (\alpha_{ij})_{n \times n}$ , then  $Ax = 0$  has nontrivial solutions iff  $\det A = 0$ .

**6.7.2 Remark** When  $\det A = 0$ , Cramer's rule cannot be used. (Why?)

### Problem Set 6.7

1. Use Cramer's rule (if applicable) to find the solutions of the following systems of linear equations :

$$\begin{array}{lll} \text{(a)} & 3x + y = 1 & \text{(b)} \quad 2x - 3y = 7 \quad \text{(c)} \quad x + 2y + 3z = 3 \\ & 5x + 2y = 3 & \quad \quad \quad x + 4y = 1 \quad \quad \quad 2x \quad \quad - z = 4 \\ & & & & \quad \quad \quad 4x + 2y + 2z = 5 \end{array}$$

$$\begin{array}{ll}
 \text{(d)} & x - y + 2z = 1 \\
 & 2x \quad \quad + 2z = 3 \\
 & 3x + y + 3z = 7 \\
 \text{(e)} & x + y + 2z = 3 \\
 & 2x + 2y + 2z = 7 \\
 & 3x + 4y + 3z = 2.
 \end{array}$$

2. Repeat Problem 1 for the systems of linear equations of Problem 1, Problem Set 5.8.
3. Find the inverse of the coefficient matrix for the systems of linear equations of Problem 1 by any method, and use Equation (4) to solve these systems.
4. If the coordinate axes in a plane are rotated through an angle  $\alpha$ , then the old coordinates  $(x, y)$  are expressed in terms of the new coordinates  $(x', y')$  as

$$\begin{aligned}
 x &= x' \cos \alpha - y' \sin \alpha \\
 y &= x' \sin \alpha + y' \cos \alpha.
 \end{aligned}$$

Use Cramer's rule to express  $(x', y')$  in terms of  $(x, y)$ .

## 6.8 EIGENVALUES, EIGENVECTORS

An  $n \times n$  real matrix has been considered as a linear transformation from  $V_n$  to  $V_n$ . The simplest type of linear transformation is the one that merely multiplies all vectors  $x$  in  $V_n$  by a fixed scalar, that is,  $Ax = \lambda x$  for some fixed  $\lambda$ . Now we shall discuss those situations where the effect of a linear transformation is simply scalar multiplication, at least in part of the domain space. In the process we shall see that determinants come in very handy. In this article we shall deal with complex vector spaces and complex scalars. Consequently, the matrices may have complex entries also.

First, we shall take up a  $3 \times 3$  matrix :

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}.$$

Our aim is to find out whether for some nonzero vectors  $x = (x_1, x_2, x_3)^T$ , it is possible to have

$$Ax = \lambda x \quad (1)$$

for some suitable scalar  $\lambda$ . From Equation (1) we have

$$Ax - \lambda x = 0,$$

or

$$Ax - \lambda(Ix) = (A - \lambda I)x = 0 \quad (2)$$

i.e.,

$$\begin{bmatrix} \alpha_{11} - \lambda & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} - \lambda & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Such an equation has nontrivial solutions, by Theorem 6.7.1, iff

$$\begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} - \lambda & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} - \lambda \end{vmatrix} = 0. \quad (3)$$

Expanding this determinant, we find that Equation (3) reduces to an algebraic equation of the third degree in  $\lambda$ . Since our scalars are complex numbers, we are assured of three roots for this equation. Let us call these roots  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$ . For each of these  $\lambda$ 's, it follows that Equation (1) has a nontrivial solution. Denote the set of these nontrivial solutions of Equation (1) for a given  $\lambda$  by  $E(\lambda)$ .

The values  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  of  $\lambda$  are called *eigenvalues* of the matrix  $A$ , and each vector belonging to  $E(\lambda)$  is called an *eigenvector* corresponding to the eigenvalue  $\lambda$ .

Corresponding to each  $\lambda_i$ ,  $i = 1, 2, 3$ , we have a set of eigenvectors, denoted by  $E(\lambda_i)$ ,  $i = 1, 2, 3$ .

The precise definition is as follows.

**6.8.1 Definition** If  $A$  is a square matrix of order  $n$ , then the values of  $\lambda$  for which the equation

$$Ax = \lambda x \quad (4)$$

has nontrivial solutions are called the *eigenvalues* of  $A$ . If  $\lambda$  is an eigenvalue, then the nonzero vectors  $x$ , for which Equation (4) holds, are called *eigenvectors* corresponding to the eigenvalue  $\lambda$ .

Eigenvalues are also called 'characteristic values' or 'proper values'. Similarly, the terms 'characteristic vectors' and 'proper vectors' are also used.

By Theorem 6.7.1, it follows that eigenvalues  $\lambda$  are simply the roots of the algebraic equation

$$\det(A - \lambda I) = 0. \quad (5)$$

Equation (5) is called the *characteristic equation* of the matrix  $A$ .  $\det(A - \lambda I)$  is a polynomial in  $\lambda$  and is called the *characteristic polynomial* of the matrix  $A$ . Note that this polynomial has degree  $n$ . Hence, there can be at most  $n$  distinct eigenvalues.

**Example 6.25** Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}.$$

To find the eigenvalues of  $A$ , we solve

$$\begin{vmatrix} 1-\lambda & -1 & 2 \\ 0 & 1-\lambda & 0 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0.$$

This gives

$$(1-\lambda)((1-\lambda)^2 + (-2)) = 0$$

or

$$(1-\lambda)(\lambda^2 - 2\lambda - 1) = 0,$$

i.e.

$$\lambda = 1, 1 + \sqrt{2}, 1 - \sqrt{2}.$$

So the eigenvalues of  $A$  are

$$\lambda_1 = 1, \lambda_2 = 1 + \sqrt{2}, \lambda_3 = 1 - \sqrt{2}.$$

To find  $E(\lambda_1) = E(1)$ , we write

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

i.e.

$$\begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$-x_2 + 2x_3 = 0, \quad x_1 + 2x_2 = 0.$$

This gives  $x_1 = -2x_2$  and  $x_2 = 2x_3$ . The eigenvectors corresponding to the eigenvalue 1 are therefore of the form  $(-4x_3, 2x_3, x_3)$ , where  $x_3 \neq 0$ . Therefore,

$$E(1) = [(-4, 2, 1)] \setminus \{0\}.$$

To find  $E(\lambda_2) = E(1 + \sqrt{2})$ , we write

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = (1 + \sqrt{2}) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} -\sqrt{2} & -1 & 2 \\ 0 & -\sqrt{2} & 0 \\ 1 & 2 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solving this, we get

$$E(1 + \sqrt{2}) = [(\sqrt{2}, 0, 1)] \setminus \{0\}.$$

Similarly,

$$E(1 - \sqrt{2}) = [(-\sqrt{2}, 0, 1)] \setminus \{0\}.$$

**6.8.2 Theorem** If  $\lambda$  is an eigenvalue of an  $n \times n$  matrix  $A$ , then the set  $K_\lambda = E(\lambda) \cup \{0\}$  is a subspace of the vector space  $V_n^C$  of  $n$ -tuples of complex numbers.

$K_\lambda$  is called the *eigenspace corresponding to  $\lambda$* . The proof is left to the reader. Note that  $K_\lambda$  is the set of all solutions of the equation  $Ax = \lambda x$ .

It is clear from Theorem 6.8.2 that if  $v$  belongs to  $E(\lambda)$ , then every non-zero scalar multiple of  $v$  is also a member of  $E(\lambda)$ . Therefore, if  $v_1 \in E(\lambda_1)$  and  $v_2 \in E(\lambda_2)$ , then  $\{v_1, v_2\}$  is LI, because neither is  $v_1$  a non-zero scalar multiple of  $v_2$  nor is  $v_2$  a nonzero scalar multiple of  $v_1$ . This idea does not stop here. It can be further extended. More precisely, we have the following result.

**6.8.3 Lemma** *Let  $A$  be a square matrix of order  $n$  having  $k$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_k$ . Let  $v_i$  be an eigenvector corresponding to the eigenvalue  $\lambda_i, i = 1, 2, \dots, k$ . Then the set  $\{v_1, v_2, \dots, v_k\}$  is LI.*

*Proof:* We shall prove this lemma by induction. The lemma is true for  $k = 1$ , i.e.  $\{v_1\}$  is LI, because  $v_1 \neq 0$ . Now suppose that it is true for  $k = r$ , i.e.  $\{v_1, v_2, \dots, v_r\}$  is LI. We shall now prove that it holds also for  $k = r + 1$ , i.e.  $\{v_1, v_2, \dots, v_r, v_{r+1}\}$  is LI. Let

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r + \alpha_{r+1} v_{r+1} = 0. \quad (6)$$

Therefore,  $A(\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r + \alpha_{r+1} v_{r+1}) = A0 = 0$ , i.e.

$$\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \dots + \alpha_r \lambda_r v_r + \alpha_{r+1} \lambda_{r+1} v_{r+1} = 0, \quad (7)$$

because  $A$  is linear and  $Av_i = \lambda_i v_i$ . Multiplying Equation (6) by  $\lambda_{r+1}$  and subtracting from Equation (7), we get

$$\alpha_1 (\lambda_1 - \lambda_{r+1}) v_1 + \alpha_2 (\lambda_2 - \lambda_{r+1}) v_2 + \dots + \alpha_r (\lambda_r - \lambda_{r+1}) v_r = 0.$$

Since  $\{v_1, v_2, \dots, v_r\}$  is LI and  $\lambda_i \neq \lambda_{r+1}, i = 1, 2, 3, \dots, r$ , we get  $\alpha_1 = 0 = \alpha_2 = \dots = \alpha_r$ . Substituting these values of  $\alpha_i$ 's in Equation (6), we get  $\alpha_{r+1} v_{r+1} = 0$ . Therefore,  $\alpha_{r+1} = 0$ , which completes the proof. ■

We shall now prove a fundamental property of eigenvalues and eigenvectors, which will be useful in the further development of the subject.

**6.8.4 Theorem** *Let  $A$  be a square matrix of order  $n$  having  $n$  distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$ . Let  $v_i$  be an eigenvector corresponding to the eigenvalue  $\lambda_i, i = 1, 2, \dots, n$ . Then the set  $\{v_1, v_2, \dots, v_n\}$  is a basis for the domain space of  $A$ . The matrix of the linear transformation  $A$  with respect to the basis  $\{v_1, v_2, \dots, v_n\}$  is*

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

*Proof:* The set  $\{v_1, v_2, \dots, v_n\}$  is LI by Lemma 6.8.3. Since the domain space is  $n$ -dimensional, it follows, by Theorem 3.6.7, that the set

$\{v_1, v_2, \dots, v_n\}$  is a basis for the domain space of  $A$ . For the second part of the theorem, note that  $Av_i = \lambda_i v_i = 0v_1 + 0v_2 + \dots + 0v_{i-1} + \lambda_i v_i + 0v_{i+1} + \dots + 0v_n$ . Thus, the  $i$ -th column of the required matrix is

$$(0, 0, \dots, \lambda_i, \dots, 0)^T.$$

$\uparrow$   
 $i$ -th place

Hence the theorem. ■

In such a case the matrix is said to have been *diagonalised*.

### Problem Set 6.8

- Determine the eigenvalues and the corresponding eigenspaces for the following matrices :

$$\begin{array}{llll}
 \text{(a)} \begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix} & \text{(b)} \begin{bmatrix} -0 & -1 \\ 1 & 0 \end{bmatrix} & \text{(c)} \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} & \text{(d)} \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix} \\
 \text{(e)} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} & \text{(f)} \begin{bmatrix} 0 & i & i \\ i & 0 & i \\ i & i & 0 \end{bmatrix} & \text{(g)} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} & .
 \end{array}$$

- Diagonalise the matrices of (b), (c), and (g) in Problem 1.
- Find the characteristic polynomial of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & i & 0 & 0 \\ 2 & \frac{1}{2} & -i & 0 \\ \frac{1}{3} & -i & \pi & -1 \end{bmatrix}.$$

Diagonalise this matrix, if possible.

- If  $\lambda$  is an eigenvalue of the matrix  $A$ , prove that
  - $\lambda^2$  is an eigenvalue of  $A^2$
  - $\lambda^n$  is an eigenvalue of  $A^n$
  - $\alpha\lambda$  is an eigenvalue of  $\alpha A$ , where  $\alpha$  is a scalar
  - $g(\lambda)$  is an eigenvalue of  $g(A)$ , where  $g$  is a polynomial.
- If  $x$  is an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda$ , prove that
  - $x$  is an eigenvector of  $A^n$  corresponding to the eigenvalue  $\lambda^n$
  - $x$  is an eigenvector of  $g(A)$  corresponding to the eigenvalue  $g(\lambda)$



6. Prove that  $\lambda = 0$  is an eigenvalue of the matrix  $A$  iff  $A$  is singular.
7. If  $\lambda$  is an eigenvalue of the matrix  $A$ , prove that
  - (a)  $\lambda$  is also an eigenvalue of  $A^T$
  - (b)  $1/\lambda$  is an eigenvalue of  $A^{-1}$ , if  $A$  is nonsingular.
8. If  $x$  is an eigenvector of  $A$  corresponding to the eigenvalue  $\lambda$ , prove that
  - (a)  $x$  need not be an eigenvector of  $A^T$  corresponding to the eigenvalue  $\lambda$
  - (b)  $x$  is an eigenvector of  $A^{-1}$  (if  $A$  is nonsingular) corresponding to the eigenvalue  $1/\lambda$ .
9. Prove that the eigenvalues of a triangular matrix are its diagonal elements.

## 6.9 WRONSKIANS

In this article we shall use determinants to discuss dependence and independence of functions. These ideas help in an understanding of the theory of ordinary linear differential equations. First, we shall define the Wronskian of functions.

**6.9.1 Definition** If  $y_1, y_2, \dots, y_n$  are  $n$  functions in  $\mathcal{C}^{(n-1)}(I)$ , then the determinant

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) & \dots & y_n(x) \\ y_1'(x) & y_2'(x) & \dots & y_n'(x) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \dots & y_n^{(n-1)}(x) \end{vmatrix}$$

is called the *Wronskian* of  $y_1, y_2, \dots, y_n$  at  $x$ , and is denoted by  $W[y_1(x), y_2(x), \dots, y_n(x)]$ .

Note that the Wronskian is a function of  $x$ .

**Example 6.26** The Wronskian of the functions  $x$  and  $\cos x$  is

$$W[x, \cos x] = \begin{vmatrix} x & \cos x \\ 1 & -\sin x \end{vmatrix} = -x \sin x - \cos x.$$

The linear dependence or independence of the functions  $y_1, y_2, \dots, y_n$  plays an important role in the solution of an ordinary linear differential equation. The following theorem tells how the dependence or independence of functions  $y_1, y_2, \dots, y_n$  is related to their Wronskian.

**6.9.2 Theorem** Let  $y_1, y_2, \dots, y_n \in \mathcal{C}^{(n)}(I)$ , and let  $W(x)$  be their Wronskian.

(a) If there exists  $x_0 \in I$  such that  $W(x_0) \neq 0$ , then  $y_1, y_2, \dots, y_n$  are LI over  $I$ .

(b) If there exists  $x_0 \in I$  such that  $W(x_0) = 0$  and if  $y_1, y_2, \dots, y_n$  are solutions of the differential equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0, \quad (1)$$

where  $a_0, a_1, \dots, a_n \in \mathcal{C}(I)$ , then  $y_1, y_2, \dots, y_n$  are LD over  $I$ .

(c) If  $y_1, y_2, \dots, y_n$  are solutions of Equation (1) and are LI over  $I$ , then  $W(x)$  never vanishes on  $I$ .

(d) If  $y_1, y_2, \dots, y_n$  are solutions of Equation (1), then either  $W(x) = 0$  over  $I$  or  $W(x)$  has no zeros in  $I$ .

*Proof:* (a) Since  $W(x_0) \neq 0$ ,  $W(x_0)$  considered as a square matrix is nonsingular by Corollary 6.5.5. Hence, by Theorem 5.5.2, the column vectors

$$v_i(x_0) = (y_i(x_0), y_i'(x_0), \dots, y_i^{(n-1)}(x_0)), i = 1, 2, \dots, n \quad (2)$$

are LI in  $V_n$ . Therefore,  $y_1, y_2, \dots, y_n$  are LI over  $I$ , by Lemma 4.9.5.

(b) If  $W(x_0) = 0$ , the columns of  $W(x_0)$ , namely,  $v_i(x_0) = (y_i(x_0), y_i'(x_0), \dots, y_i^{(n-1)}(x_0)), i = 1, \dots, n$ , are LD in  $V_n$ , by Theorem 5.5.2. Therefore, there exist scalars  $C_1, C_2, \dots, C_n$ , not all zero, such that

$$C_1 v_1(x_0) + C_2 v_2(x_0) + \dots + C_n v_n(x_0) = 0.$$

This means

$$C_1 y_1^{(k)}(x_0) + C_2 y_2^{(k)}(x_0) + \dots + C_n y_n^{(k)}(x_0) = 0, k = 0, 1, 2, \dots, (n-1), \quad (3)$$

for at least one scalar  $C_k$  not zero. Consider the linear combination  $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ . Since  $y_1, y_2, \dots, y_n$  are solutions of Equation (1),  $y$  is also a solution of (1). But by Equation (3), we have

$$y(x_0) = y'(x_0) = \dots = y^{(n-1)}(x_0) = 0. \quad (4)$$

Thus,  $y$  is a solution of Equation (1) and satisfies the initial conditions (4). But the zero function also satisfies Equation (1) and the initial conditions (4). Hence, by Theorem 4.9.2 (existence and uniqueness),  $y(x) = 0$  for all  $x$  in  $I$ .

Thus,  $C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x) = 0$  for all  $x \in I$  and at least one  $C_k \neq 0$ . Therefore,  $y_1, y_2, \dots, y_n$  are LD over  $I$ .

(c) This is the contrapositive statement of (b).

(d) It follows from (b) and (c). ■

**6.9.3 Theorem** If  $y_1, y_2$  are solutions of

$$a_0 y'' + a_1 y' + a_2 y = 0, \quad (5)$$

where  $a_0, a_1$ , and  $a_2$  belong to  $\mathcal{C}(I)$ , and  $a_0$  is nowhere zero on  $I$ , then

$$W[y_1(x), y_2(x)] = Ce^{-\int \frac{a_1(x)}{a_0(x)} dx} \quad (6)$$

*Proof:*  $y_1$  and  $y_2$  are solutions of (5). Therefore,

$$\begin{aligned} & a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \\ \text{and} \quad & a_0 y_2'' + a_1 y_2' + a_2 y_2 = 0. \end{aligned}$$

$$\text{Also} \quad W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2.$$

$$\begin{aligned} \text{So} \quad \frac{dW}{dx} &= y_1' y_2 + y_1 y_2' - y_1'' y_2 - y_1' y_2' = y_1 y_2' - y_1' y_2 \\ &= y_1 \left( -\frac{a_1}{a_0} y_2' - \frac{a_2}{a_0} y_2 \right) - y_2' \left( -\frac{a_1}{a_0} y_1' - \frac{a_2}{a_0} y_1 \right) \\ &= -\frac{a_1}{a_0} (y_2 y_1' - y_1 y_2') = -\frac{a_1}{a_0} W. \end{aligned}$$

Hence, on integration, we obtain

$$W[y_1(x), y_2(x)] = C e^{-\int \frac{a_1(x)}{a_0(x)} dx},$$

where  $C$  is an arbitrary constant. ■

**6.9.4 Remark** The foregoing result can be extended to a differential equation of the  $n$ -th order. Here we state the general result. If  $y_1, y_2, \dots, y_n$  are  $n$  solutions of the  $n$ -th order normal linear differential equation  $a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y' = 0$ , on an interval  $I$ ,

$$\text{then } W[y_1(x), y_2(x), \dots, y_n(x)] = C e^{-\int \frac{a_1(x)}{a_0(x)} dx} \quad (7)$$

Equations (6) and (7) are known as *Abel's formulae*.

### Problem Set 6.9

- Use the Wronskian to prove that the following sets of functions are LI over  $I$ :
  - $e^{ax}, e^{bx}$ ;  $a \neq b$ ;  $I$  any interval
  - $e^{m_1 x}, e^{m_2 x}, \dots, e^{m_n x}$ ;  $I$  any interval  
 $m_1, m_2, \dots, m_n$  all distinct
  - $1, x, x^2$ ;  $I$  any interval
  - $1, x, x^2, \dots, x^{n-1}$ ;  $I$  any interval
  - $e^x, \sin 2x$ ;  $I$  any interval
  - $e^{\alpha x}, x e^{\alpha x}$ ;  $I$  any interval
  - $e^{\alpha x} \sin \beta x, e^{\alpha x} \cos \beta x$ ;  $I$  any interval
  - $\ln x, x \ln x$ ;  $I = (0, \infty)$ .
- Use Abel's formula to find the Wronskian of the linearly independent solutions of the following normal linear differential equations:
  - $y'' - 4y' + 4y = 0$
  - $y'' + 2xy = 0$

- (c)  $(1 - x^2)y'' - 2xy' + 2y = 0$ ;  $y_1(0) = y_1'(0) = 2$ ,  
 $y_2(0) = -y_2'(0) = 1$   
 (d)  $x^2y'' - 3xy' + y = 0$ ;  $y_1(-1) = y_1'(-1) = 2$ ,  $y_2(-1) = 0$ ,  
 $y_2'(-1) = -1$ .

3. What is the value of  $C$  in Equation (7) if  $y_1, y_2, \dots, y_n$  are LD?

## 6.10 CROSS PRODUCT IN $V_3$

In this article we shall discuss the product of vectors in  $V_3$ , mentioned in § 2.3.

**6.10.1 Definition** If  $u = (a_1, a_2, a_3)$  and  $v = (b_1, b_2, b_3)$  are two vectors in  $V_3$ , their *cross product*, written as  $u \times v$ , is defined as the vector

$$u \times v = \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right).$$

Since the cross product is a vector, it is also called *vector product*.

In simplified form Definition 6.10.1 says

$$(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1).$$

If we use the unit vectors  $i, j, k$ , we obtain

$$(a_1i + a_2j + a_3k) \times (b_1i + b_2j + b_3k) \\ = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i + \begin{vmatrix} a_3 & a_1 \\ b_3 & b_1 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k,$$

which is easily remembered in the determinant form

$$\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}. \quad (1)$$

To obtain the cross product we expand (1) in terms of the first row as if it were a determinant. Note that (1) is not a 'pukka' determinant.

From Definition 6.10.1 it follows immediately that

$$i \times j = k, \quad j \times k = i, \quad k \times i = j,$$

$$j \times i = -k, \quad k \times j = -i, \quad i \times k = -j,$$

$$\text{and } i \times i = 0 = j \times j = k \times k.$$

Note that the cross product is a binary operation on  $V_3$ .

**Example 6.27**  $(1, 2, -1) \times (0, -1, 3)$

$$= \left( \begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 3 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \right) \\ = (5, -3, -1).$$

**6.10.2 Theorem (Properties of cross product)** Let  $u, v, w$  be any three vectors in  $V_3$ . Then

(a)  $u \times v = -(v \times u)$ .

(b)  $u \times (v + w) = u \times v + u \times w$ .

(c) For  $\alpha \in R$ ,  $\alpha u \times v = \alpha(u \times v) = u \times \alpha v$ .

(d)  $u \cdot (u \times v) = 0$ , and  $v \cdot (u \times v) = 0$ , i.e.  $u \times v$  is orthogonal to both  $u$  and  $v$ .

(e)  $|u \times v|^2 = |u|^2 |v|^2 - |u \cdot v|^2$ .

Since the proofs are straightforward, we shall prove only (e).

*Proof of (e):* Let  $u = (a_1, a_2, a_3)$  and  $v = (b_1, b_2, b_3)$ . Then

$$\begin{aligned} |u|^2 |v|^2 - |u \cdot v|^2 &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1 b_1 + a_2 b_2 + a_3 b_3)^2 \\ &= \sum a_i^2 (b_i^2 + b_j^2 + b_k^2) - 2 \sum a_i b_i a_j b_j \\ &= \sum (a_i^2 b_j^2 + a_j^2 b_i^2) - 2 \sum a_i b_i a_j b_j \\ &= \sum (a_i b_j - a_j b_i)^2 \\ &= |u \times v|^2. \quad \blacksquare \end{aligned}$$

**6.10.3 Corollary** If  $\theta$  is the angle between two nonzero vectors  $u$  and  $v$  in  $V_3$ , then

$$|u \times v| = |u| |v| \sin \theta.$$

*Proof:* We have

$$u \cdot v = |u| |v| \cos \theta \quad (\text{cf Definition 2.3.3}).$$

Hence, from (e) of Theorem 6.10.2 it follows that

$$\begin{aligned} |u \times v|^2 &= |u|^2 |v|^2 - |u \cdot v|^2 \\ &= |u|^2 |v|^2 (1 - \cos^2 \theta) \\ &= |u|^2 |v|^2 \sin^2 \theta. \end{aligned}$$

So  $|u \times v| = |u| |v| \sin \theta$ , since  $0 < \theta < \pi$ .  $\blacksquare$

Note that  $|u| |v| \sin \theta$  is the area of the parallelogram formed by vectors  $u$  and  $v$ . Hence,  $|u \times v|$  represents the area of the parallelogram formed by  $u$  and  $v$ .

#### DIRECTION OF $u \times v$

Given two vectors  $u$  and  $v$ , we have just proved that

(i)  $|u \times v| = |u| |v| \sin \theta$ , where  $\theta$  is the angle between  $u$  and  $v$ , and

(ii)  $u \times v$  is perpendicular to both  $u$  and  $v$  and therefore to the plane containing  $u$  and  $v$ .

(i) specifies the magnitude of  $u \times v$ , but (ii) leaves us to decide which of the two directions perpendicular to the plane of  $u$  and  $v$  is the direction of  $u \times v$ . We shall now settle this question.

We start from the ordered triple of vectors  $(i, j, k)$ . Any ordered triple of linearly independent vectors  $(u, v, w)$  is said to be a *positive triple* if, by gradually changing the directions of  $u, v, w$  but without making them

*linearly dependent* (i.e. without making any of the vectors cross the plane of the other two), we can bring them into coincidence with the directions of  $i, j$ , and  $k$ . Otherwise it is said to be a *negative triple*.

Examples of positive triples are

$$(i, j, k), \quad (j, k, i), \quad (k, i, j).$$

Examples of negative triples are

$$(j, i, k), \quad (k, j, i), \quad (i, k, j).$$

**6.10.4 Lemma** Three linearly independent vectors  $u = (a_1, a_2, a_3)$ ,  $v = (b_1, b_2, b_3)$ ,  $w = (c_1, c_2, c_3)$  form a positive triple iff

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} > 0.$$

*Proof:* A positive triple, which actually coincides with the directions of  $i, j, k$ , is of the form

$$u = ai, \quad v = bj, \quad w = ck,$$

where  $a, b$ , and  $c$  are positive. So the determinant of the lemma is

$$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

which is positive. Now any other positive triple is obtained by gradually changing the directions of  $u, v$ , and  $w$ , but without ever making them LD. This process continuously changes the value of the aforesaid determinant, but without ever making the determinant take the value zero (since such a value will correspond to making  $u, v$ , and  $w$  LD). Hence, every positive triple has a positive value for this determinant and vice versa. ■

Now we can see that  $u \times v$ , as defined, has that direction perpendicular to the plane of  $u, v$  that makes  $u, v, u \times v$  a positive triple; for the determinant of Lemma 6.10.4 now becomes

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & a_1b_2 - a_2b_1 \end{vmatrix},$$

whose value is  $\Sigma (a_2b_3 - a_3b_2)^2$ , which is positive. Thus,  $u \times v$  is given by Figure 6.1.

In other words, if the angle between  $u$  and  $v$  is  $\theta$ ,  $0 < \theta < \pi$ , then the direction of  $u \times v$  is the one in which a right-handed screw would move when rotated through the angle  $\theta$  from  $u$  to  $v$ .

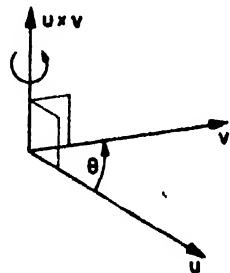


FIGURE 6.1

**6.10.5 Definition** If  $u$ ,  $v$ , and  $w$  are three vectors in  $V_3$ ,  $u \cdot (v \times w)$  is called the *scalar triple product* of  $u$ ,  $v$ , and  $w$ .

Note that, even if we remove the parentheses in Definition 6.10.5 and write  $u \cdot v \times w$  in place of  $u \cdot (v \times w)$ , there cannot arise any ambiguity, since  $(u \cdot v) \times w$  is meaningless. (Why?)

**6.10.6 Theorem (Properties of scalar triple product)** Let  $u = (a_1, a_2, a_3)$ ,  $v = (b_1, b_2, b_3)$ , and  $w = (c_1, c_2, c_3)$  be three vectors in  $V_3$ . Then

$$(a) \quad u \cdot v \times w = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

$$(b) \quad u \cdot v \times w = u \times v \cdot w = v \cdot w \times u = v \times w \cdot u \\ = w \cdot u \times v = w \times u \cdot v.$$

(c)  $|u \cdot v \times w|$  represents the volume of the parallelepiped formed by the vectors  $u$ ,  $v$ , and  $w$ .

*Proof:* (a) Using the definitions of dot and cross product, we have  $u \cdot v \times w$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (\text{by expansion of the determinant in terms of the first row}).$$

$$(b) \quad u \cdot v \times w = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ = \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \quad (\text{by two interchanges of rows}) \\ = v \cdot w \times u.$$

Similarly,  $v \cdot w \times u = w \cdot u \times v$ .

Thus,  $u \cdot v \times w = v \cdot w \times u = w \cdot u \times v$ .

Since the dot product is commutative, these are also equal to  $v \times w \cdot u = w \times u \cdot v = u \times v \cdot w$ .

(c) If one of the vectors is zero or  $v \times w = 0$ , the proof is obvious. We therefore assume that none of these vectors is zero and so also  $v \times w \neq 0$ .

Hence, the vector  $v \times w$  is perpendicular to the plane determined by the vectors  $v$  and  $w$  (cf Theorem 6.10.2 (d)). So either  $v \times w$  or  $-(v \times w)$  makes an angle  $\phi < \pi/2$  with  $u$ , as shown in Figure 6.2.

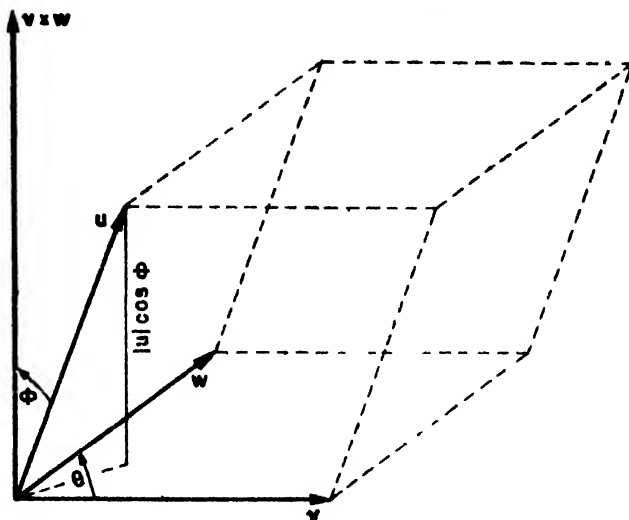


FIGURE 6.2

$$\begin{aligned}
 \text{So } |u \cdot v \times w| &= |u| |v \times w| \cos \phi \\
 &= (|u| \cos \phi) |v \times w| \\
 &= (\text{height of the parallelepiped formed by } u, v, \text{ and } w) \times (\text{area of the parallelogram formed by } v \text{ and } w) \\
 &= \text{volume of the parallelepiped formed by } u, v, \text{ and } w. \quad \blacksquare
 \end{aligned}$$

**6.10.7 Corollary**  $u \cdot v \times w = 0$  iff  $u, v$ , and  $w$  are LD.

*Proof:* From the proof of part (c) of Theorem 6.10.6, it follows that

$$\begin{aligned}
 |u \cdot v \times w| &= |u| |v \times w| \cos \phi \\
 &= |u| |v| |w| \sin \theta \cos \phi.
 \end{aligned}$$

Hence,  $u \cdot v \times w = 0$  means one of the three vectors is 0 or  $\sin \theta = 0$  or  $\cos \phi = 0$ . If one of the vectors is 0, then  $u, v$  and  $w$  are LD. If  $\sin \theta = 0$  or  $\cos \phi = 0$ , then the three vectors are coplanar and hence LD. So  $u \cdot v \times w = 0$  implies that the three vectors are LD. The converse is obvious.  $\blacksquare$

We shall now consider two important applications of the cross product to geometry.

(i) *Vector equation of a plane through three given points  $P_1, P_2, P_3$  in space:* Let  $Q$  be a point in the plane  $P_1P_2P_3$  (see Figure 6.3). The vectors



$\overrightarrow{P_1P_2}$  and  $\overrightarrow{P_1P_3}$  lie in the plane. The problem now reduces to finding the

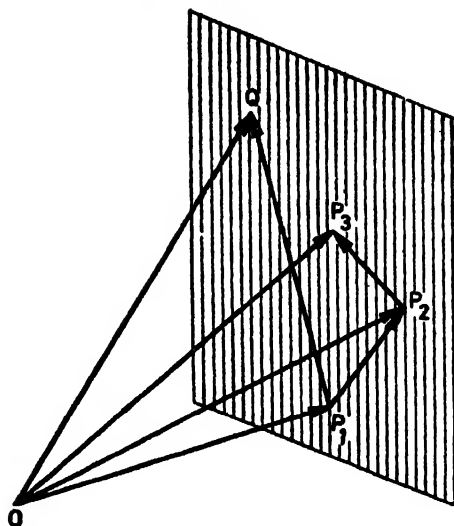


FIGURE 6.3

equation of a plane through  $P_1$  and perpendicular to the vector  $N = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ . Hence, the required equation is  $\overrightarrow{P_1Q} \cdot N = 0$ ,

$$\text{i.e.} \quad \overrightarrow{P_1Q} \cdot (\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}) = 0. \quad (2)$$

This equation is satisfied by all points  $Q$  on the required plane and by no points off the plane.

If  $P_1$ ,  $P_2$ , and  $P_3$  are  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ , and  $(x_3, y_3, z_3)$  respectively, then Equation (2) becomes

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

**Example 6.28** The equation of the plane through  $P_1(0, 0, 0)$ ,  $P_2(1, -1, 1)$ , and  $P_3(2, 1, 2)$  is

$$\begin{vmatrix} x & y & z \\ 1 & -1 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 0,$$

i.e.  $x - z = 0$ .

(ii) **Shortest distance between two lines**: If we have two lines in a plane, then either they are parallel or they intersect. On the other hand, two lines in space can behave in any one of three ways: (a) they are

parallel or (b) they intersect or (c) neither are they parallel nor do they intersect. In case (c) they are called *skew lines*. Let  $L_1$  and  $L_2$  be two skew lines (see Figure 6.4).

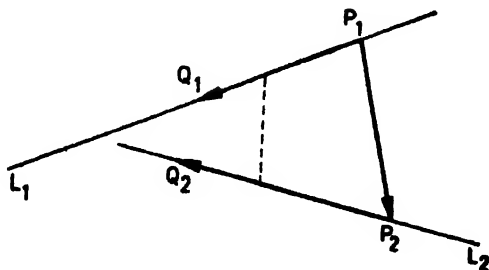


FIGURE 6.4

Let  $P_1, Q_1$  be two points on  $L_1$ , and  $P_2, Q_2$  two points on  $L_2$ . The vector  $\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}$  is perpendicular to both  $L_1$  and  $L_2$ . The scalar projection of  $\overrightarrow{P_1P_2}$  on this perpendicular is the required shortest distance between  $L_1$  and  $L_2$ . It is given by

$$\frac{|\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}|}{|\overrightarrow{P_1Q_1} \times \overrightarrow{P_2Q_2}|}.$$

**Example 6.29** Let the two lines be

$$L_1: \frac{x-1}{3} = \frac{y-2}{2} = \frac{z-1}{5} \text{ and } L_2: \frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2}.$$

Since  $L_1$  is parallel to the vector  $(3, 2, 5)$ , and  $L_2$  is parallel to the vector  $(1, 2, 2)$ , the vector  $N = (3, 2, 5) \times (1, 2, 2)$  is perpendicular to both  $L_1$  and  $L_2$ . We can take  $P_1$  as  $(1, 2, 1)$  and  $P_2$  as  $(3, 2, -4)$ . Hence, the shortest distance between  $L_1$  and  $L_2$  is

$$\begin{aligned} & \frac{|(2, 0, -5) \cdot (3, 2, 5) \times (1, 2, 2)|}{|(3, 2, 5) \times (1, 2, 2)|} \\ &= \left| \begin{vmatrix} 2 & 0 & -5 \\ 3 & 2 & 5 \\ 1 & 2 & 2 \end{vmatrix} \right| \div \sqrt{36 + 1 + 16} \\ &= |-12 - 20| \div \sqrt{53} \\ &= \frac{32}{\sqrt{53}}. \end{aligned}$$

**6.10.8 Remark** The method in Example 6.29 fails in case the lines  $L_1$  and  $L_2$  are parallel. (Why?) In this case let  $P_1$  and  $Q_1$  be two points

on  $L_1$ , and  $P_2$  a point on  $L_2$  (Figure 6.5). Then the scalar projection of  $\overrightarrow{P_1P_2}$  on  $L_1$  gives

$$d = \frac{|\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1}|}{|\overrightarrow{P_1Q_1}|}.$$

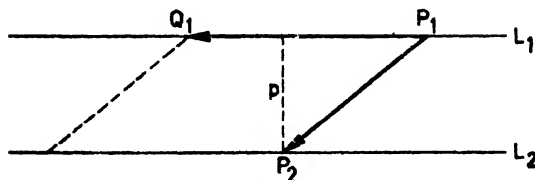


FIGURE 6.5

Hence, the (shortest) distance  $p$  between  $L_1$  and  $L_2$  is given by

$$\begin{aligned} p &= \sqrt{|\overrightarrow{P_1P_2}|^2 - d^2} \\ &= \sqrt{|\overrightarrow{P_1P_2}|^2 - \frac{|\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q_1}|^2}{|\overrightarrow{P_1Q_1}|^2}} \\ &= \frac{|\overrightarrow{P_1P_2} \times \overrightarrow{P_1Q_1}|}{|\overrightarrow{P_1Q_1}|}. \quad (\text{cf Theorem 6.10.2 (e)}) \end{aligned}$$

### Problem Set 6.10

- Evaluate  $u \times v$  for the vectors  $u$  and  $v$  in the following :
  - $u = i + j + k, v = 2i - 3j + k$
  - $u = i - 2k, v = 2j + 7k$
  - $u = (1, 3, 0), v = (2, -1, -1)$
  - $u = (-1, 2, 3), v = (1, 1, 2)$ .
- Prove that  $0 \times v = 0$  for every vector  $v$  in  $V_3$ .
- Prove the distributive law  $u \times (v + w) = (u \times v) + (u \times w)$  and verify it for the vectors  $u, v$ , and  $w$  in the following :
  - $u = i - 2j + k, v = 3i - j, w = 2i + j - 4k$
  - $u = (1, 2, 3), v = (3, 0, 1), w = (2, 4, 5)$
  - $u = (-1, 1, 2), v = (1, -1, -1), w = (4, 2, -3)$ .
- In the discussion of direction of the vector  $u \times v$  we have taken  $0 < \theta < \pi$ . What happens when  $\theta = 0$  or  $\theta = \pi$ ?
- Determine whether the vectors  $u \times v$  and  $w$  lie on the same side of

the plane containing the vectors  $u$  and  $v$  for the vectors  $u$ ,  $v$ , and  $w$  in Problem 3.

6. Find the equation of the plane passing through the three points

(a)  $A(1, 2, -3)$ ,  $B(3, -1, 1)$ , and  $C(2, 1, 1)$

(b)  $A(1, 2, 3)$ ,  $B(3, 0, 1)$ , and  $C(2, 4, 5)$

(c)  $A(1/2, 1, -1)$ ,  $B(1/2, -1, -1)$ , and  $C(4, 1/3, -3)$ .

7. Prove that for vectors  $u$ ,  $v$ , and  $w$  of  $V_3$

(a)  $(u \times v) \times w = (u \cdot w)v - (v \cdot w)u$

(b)  $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

8. Evaluate the product

$$(a \times b) \times (c \times d)$$

of the four vectors  $a$ ,  $b$ ,  $c$ , and  $d$  of  $V_3$ . Prove that this is a vector in the plane of  $a$  and  $b$  as well as of  $c$  and  $d$ .

9. Find the shortest distance between the skew lines

$$\frac{x-2}{3} = \frac{y+1}{2} = 1-z \text{ and } x = y = z.$$

10. Find the equation of the plane containing the line

$$\frac{x-2}{3} = \frac{y+1}{2} = 1-z$$

and parallel to the line  $x = y = z$ .

11. Show that the line

$$L: \frac{x+1}{2} = \frac{y-1}{3} = \frac{z+2}{4}$$

is parallel to the plane

$$P: 2x - 4y + 2z = 3,$$

and find the equation of a line lying in the plane  $P$  and perpendicular to the line  $L$ .

12. Find a plane through  $P(1, 2, 3)$  and perpendicular to the line of intersection of the planes  $2x + 3y - z = 1$  and  $3x - y - z = 4$ .
13. Find a plane through the points  $P(1, -1, 2)$  and  $Q(2, 0, 1)$  and perpendicular to the plane  $3x - 4y + z = 0$ .

## Chapter 7

# More Matrix Theory

In this chapter we shall discuss some miscellaneous topics to give the reader a glimpse of what lies beyond. The intention is not to give an exhaustive coverage but to further our knowledge by giving a brief account of the following topics :

- (i) Similarity of matrices;
- (ii) Inner product spaces;
- (iii) Orthogonal and unitary matrices; and
- (iv) Application to reduction of quadrics.

## 7.1 SIMILARITY OF MATRICES

The central idea here is to find out the effect of working with different bases in a finite-dimensional vector space  $V$ . Let two ordered bases

$$F = \{u_1, u_2, \dots, u_n\} \text{ and } G = \{v_1, v_2, \dots, v_n\}$$

be given in  $V$ . We pose the following questions.

**Question 1** Given a vector  $x \in V$ , what is the relationship between its coordinate vectors  $[x]_F$  and  $[x]_G$ ? (Recall Definition 3.6.10.)

**Question 2** Given a linear operator  $T: V \rightarrow V$ , it has different matrices when referred to the different bases  $F$  and  $G$ . What is the relation between these matrices, namely,  $(T: F, F)$  and  $(T: G, G)$ ?

**Question 3** Given a coordinate vector  $(\alpha_1, \alpha_2, \dots, \alpha_n)$ , we can construct two different vectors

$$x = \sum_{i=1}^n \alpha_i u_i \text{ and } y = \sum_{i=1}^n \alpha_i v_i$$

in  $V$ . What is the relation between  $x$  and  $y$ ?

**Question 4** Given an  $n \times n$  matrix  $(c_{ij})$  of scalars, we can define two linear operators  $R$  and  $S$  by

$$R(u_j) = \sum_{i=1}^n c_{ij} u_i \text{ and } S(v_j) = \sum_{i=1}^n c_{ij} v_i, \quad j = 1, 2, \dots, n.$$

What is the relation between  $R$  and  $S$ ?

Essentially, these four questions can be unified by asking for the relation between

- (i) two coordinate vectors arising from the same element  $x$  of  $V$ ;
- (ii) two matrices arising from the same linear operator on  $V$ ;
- (iii) two vectors in  $V$  arising from the same coordinate vector; and
- (iv) two linear operators on  $V$  arising from the same matrix.

In order to answer these questions we construct the important linear operator  $A$  on  $V$ , which transforms the basis  $F$  into the basis  $G$ . In fact,  $A: V \rightarrow V$  is defined by  $A(u_i) = v_i$ ,  $i = 1, 2, \dots, n$ . Denoting the matrix  $(A: F, F)$  by  $(\alpha_{ij})$ , we can write the definition of  $A$  as

$$v_j = A(u_j) = \sum_{i=1}^n \alpha_{ij} u_i, \quad j = 1, 2, \dots, n. \quad (1)$$

Note that  $A$  is one-one, since, if  $x = \sum_{i=1}^n \alpha_i u_i$ , then  $Ax = 0$  gives

$$0 = A\left(\sum_{i=1}^n \alpha_i u_i\right) = \sum_{i=1}^n \alpha_i A(u_i) = \sum_{i=1}^n \alpha_i v_i$$

or  $\alpha_i = 0$  for all  $i = 1, 2, \dots, n$ , i.e.  $x = 0$ . Thus,  $A$  is a nonsingular linear transformation and so  $(\alpha_{ij})$  is a nonsingular matrix. This matrix is called *the matrix of change of basis from the  $F$ -basis to the  $G$ -basis*.

These four questions can now be respectively answered by the following four theorems.

**7.1.1 Theorem** Let  $x \in V$ . Let  $[x]_F$  and  $[x]_G$  be the coordinate vectors, written as column vectors, of  $x$ , relative to the bases  $F$  and  $G$ , respectively. Then

$$[x]_F = (\alpha_{ij})[x]_G, \quad (2)$$

where the matrix  $(\alpha_{ij})$  is defined by Equation (1).

**Proof:** Let

$$[x]_F = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \text{ and } [x]_G = (\beta_1, \beta_2, \dots, \beta_n)^T.$$

$$\begin{aligned} \text{Then } \sum_i \alpha_i u_i = x &= \sum_j \beta_j v_j = \sum_j \beta_j A(u_j) \\ &= \sum_j \beta_j \sum_i \alpha_{ij} u_i = \sum_i \left( \sum_j \beta_j \alpha_{ij} \right) u_i. \end{aligned}$$

Since the  $u_i$ 's are LI, it follows that

$$\alpha_i = \sum_j \alpha_{ij} \beta_j, \quad i = 1, 2, \dots, n,$$

i.e.

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = (x_{ij}) \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}.$$

This proves the theorem. ■

**7.1.2 Theorem** Let  $T$  be a linear operator on  $V$ . Let  $(T : F, F)$  be the matrix  $B = (\beta_{ij})$  and  $(T : G, G)$  be the matrix  $C = (\gamma_{ij})$ . Then, with  $(\alpha_{ij})$  defined as in Equation (1),

$$B(\alpha_{ij}) = (\alpha_{ij})C. \quad (3)$$

In other words,

$$B = (\alpha_{ij})C(\alpha_{ij})^{-1}. \quad (4)$$

*Proof:* We have

$$T(u_i) = \sum_j \beta_{ij} u_j, \text{ and } T(v_j) = \sum_i \gamma_{ij} v_i, \quad j = 1, 2, \dots, n.$$

Now, for each  $j = 1, 2, \dots, n$ , we have

$$\begin{aligned} T(v_j) &= T(A(u_i)) , & (\text{where } A \text{ is defined by (1)}) \\ &= T(\sum_k \alpha_{kj} u_k) , & (\text{again by (1)}) \\ &= \sum_k \alpha_{kj} T(u_k) = \sum_k \alpha_{kj} (\sum_i \beta_{ik} u_i) \\ &= \sum_i (\sum_k \beta_{ik} \alpha_{kj}) u_i . \end{aligned} \quad (5)$$

On the other hand, for each  $j = 1, 2, \dots, n$ ,

$$\begin{aligned} T(v_j) &= \sum_k \gamma_{kj} v_k = \sum_k \gamma_{kj} A(u_i) \\ &= \sum_k \gamma_{kj} \sum_i \alpha_{ik} u_i & (\text{by (1)}) \\ &= \sum_i (\sum_k \alpha_{ik} \gamma_{kj}) u_i . \end{aligned} \quad (6)$$

Equating (5) and (6) and observing that the  $u_i$ 's are LI, we get, for each  $i, j = 1, 2, \dots, n$ ,

$$\sum_k \beta_{ik} \alpha_{kj} = \sum_k \alpha_{ik} \gamma_{kj}.$$

This gives Equation (3) and proves the theorem. ■

**7.1.3 Theorem** If  $(\alpha_1, \alpha_2, \dots, \alpha_n)$  is a given coordinate vector and  $x = \sum_i \alpha_i u_i$ ,  $y = \sum_i \alpha_i v_i$ , then the two vectors  $x$  and  $y$  are connected by the relation

$$Ax = y, \quad (7)$$

where  $A$  is defined by Equation (1).

*Proof:* Since  $A(u_i) = v_i$  for all  $i = 1, 2, \dots, n$ , it follows that

$$Ax = A(\sum_i \alpha_i u_i) = \sum_i \alpha_i A(u_i) = \sum_i \alpha_i v_i = y. \quad \blacksquare$$

**7.1.4 Theorem** Let  $(c_{ij})$  be a given  $n \times n$  matrix. Define linear operators  $R$  and  $S$  by

$$R(u_j) = \sum_i c_{ij} u_i \text{ and } S(v_j) = \sum_i c_{ij} v_i, \quad j = 1, 2, \dots, n.$$

Then

$$SA = AR. \quad (8)$$

In other words,

$$S = ARA^{-1}, \quad (9)$$

where  $A$  is defined by (1).

*Proof:* We have, for each  $j = 1, 2, \dots, n$ ,

$$\begin{aligned} (SA)(u_j) &= S(A(u_j)) = S(v_j) \\ &= \sum_i c_{ij} v_i = \sum_i c_{ij} A(u_i) = A(\sum_i c_{ij} u_i) \\ &= A(R(u_j)) = (AR)(u_j). \end{aligned}$$

Hence

$$SA = AR. \quad \blacksquare$$

Equations (3), (4), (8), and (9) provide the motivation for the following definition.

**7.1.5 Definition** (a) Two  $n \times n$  matrices  $B$  and  $C$  are said to be *similar* if there exists a nonsingular matrix  $P$  such that  $BP = PC$  or  $B = PCP^{-1}$ .

(b) Two linear operators  $S$  and  $R$  on  $V$  are said to be *similar* if there exists a nonsingular linear operator  $A$  on  $V$  such that  $SA = AR$  or  $S = ARA^{-1}$ .

In the light of this definition the answers to Questions 2 and 4 can now be briefly stated as: The matrices or the linear transformations in question should be similar.

Let us now look at a comprehensive numerical example, which illustrates Theorems 7.1.1 to 7.1.4.

**Example 7.1** Let  $V = V_3$ . Consider the ordered bases

$$F = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$$

and

$$G = \{(-1, -2, 1), (1, 3, -1), (0, -1, 2)\}.$$

(a) Given  $x = (1, 1, 1)$ , verify result (2).

(b) Given a linear operator  $T: V_3 \rightarrow V_3$  defined by  $T(x_1, x_2, x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, x_2 - x_1)$ , verify result (3).

(c) Given the coordinate vector  $(4, 5, -1)$ , verify result (7).

(d) Given the matrix

$$(c_{ij}) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ -1 & -2 & -1 \end{bmatrix},$$



verify result (8).

First, we have to construct the matrix  $(\alpha_{ij})$  defined in Equation (1), corresponding to the linear transformation  $A: V_3 \rightarrow V_3$  given by

$$A(0, 1, 1) = (-1, -2, 1), \quad A(1, 0, 1) = (1, 3, -1), \quad A(1, 1, 0) = (0, -1, 2).$$

So we have to find  $\alpha_{ij}$ 's such that

$$\alpha_{11}(0, 1, 1) + \alpha_{21}(1, 0, 1) + \alpha_{31}(1, 1, 0) = (-1, -2, 1)$$

$$\alpha_{12}(0, 1, 1) + \alpha_{22}(1, 0, 1) + \alpha_{32}(1, 1, 0) = (1, 3, -1)$$

$$\alpha_{13}(0, 1, 1) + \alpha_{23}(1, 0, 1) + \alpha_{33}(1, 1, 0) = (0, -1, 2).$$

In other words, we have to solve the following three systems, each of three equations in three unknowns:

$$\begin{array}{rcl} 0\alpha + \beta + \gamma & = & -1 \\ \alpha + 0\beta + \gamma & = & -2 \\ \alpha + \beta + 0\gamma & = & 1 \end{array} \quad \left| \begin{array}{c} 1 \\ 3 \\ -1 \end{array} \right| \quad \begin{array}{c} 0 \\ -1 \\ 2 \end{array}$$

The matrix for these three systems is

$$\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & -2 & 3 & -1 \\ 1 & 1 & 0 & 1 & -1 & 2 \end{bmatrix}.$$

Applying the row reduction process to this matrix, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 & -2 & \frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

So

$$(\alpha_{ij}) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{3}{2} & \frac{3}{2} \\ -2 & \frac{1}{2} & -\frac{3}{2} \end{bmatrix}.$$

(a) Let  $[x]_F$  be  $(\alpha_1, \alpha_2, \alpha_3)^T$ . Then

$$(1, 1, 1) = \alpha_1(0, 1, 1) + \alpha_2(1, 0, 1) + \alpha_3(1, 1, 0).$$

From this we get  $\alpha_1 = 1/2 = \alpha_2 = \alpha_3$ .

Let  $[x]_G = (\beta_1, \beta_2, \beta_3)^T$ . Then

$$(1, 1, 1) = \beta_1(-1, -2, 1) + \beta_2(1, 3, -1) + \beta_3(0, -1, 2),$$

which gives  $\beta_1 = -1, \beta_2 = 0, \beta_3 = 1$ . Thus,

$$[x]_F = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T \text{ and } [x]_G = (-1, 0, 1)^T.$$

We have now only to check

$$\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & -\frac{2}{3} & \frac{2}{3} \\ -2 & \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

which is easily verified.

(b) We have to compute the matrix  $B = (T: G, G) = (\beta_{ij})$ . First, observe that, from the hypothesis,

$T(0, 1, 1) = (1, 3, 1)$ ,  $T(1, 0, 1) = (2, 2, -1)$ , and  $T(1, 1, 0) = (1, 3, 0)$ .

The scalars  $\beta_{ij}$ 's then satisfy

$$\begin{aligned} (1, 3, 1) &= \beta_{11}(0, 1, 1) + \beta_{21}(1, 0, 1) + \beta_{31}(1, 1, 0) \\ (2, 2, -1) &= \beta_{12}(0, 1, 1) + \beta_{22}(1, 0, 1) + \beta_{32}(1, 1, 0) \\ (1, 3, 0) &= \beta_{13}(0, 1, 1) + \beta_{23}(1, 0, 1) + \beta_{33}(1, 1, 0). \end{aligned}$$

To find the  $\beta_{ij}$ 's we have to solve three systems, each of three equations in three unknowns. Setting up the matrix for the row reduction process, we get

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 3 & 2 & 3 \\ 1 & 1 & 0 & 1 & -1 & 0 \end{bmatrix}$$

which, on row reduction, finally leads to

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & -1 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{5}{3} & 2 \end{bmatrix}.$$

Hence,

$$B = (\beta_{ij}) = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 1 \\ -\frac{1}{3} & -\frac{1}{3} & -1 \\ \frac{2}{3} & \frac{5}{3} & 2 \end{bmatrix}.$$

Again, to compute the matrix  $C = (T: G, G) = (\gamma_{ij})$ , observe that  $T(-1, -2, 1) = (0, -4, -1)$ ,  $T(1, 3, -1) = (0, 6, 2)$ , and  $T(0, -1, 2) = (2, 0, -1)$ .

The equations to be solved here are

$$\begin{array}{rcl} -\alpha + \beta & = & 0 \\ -2\alpha + 3\beta - \gamma & = & -4 \\ \alpha - \beta + 2\gamma & = & -1 \end{array} \quad \begin{array}{c|c} 0 & 2 \\ 6 & 0 \\ 2 & -1 \end{array}.$$

The matrix of this system is

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 2 \\ -2 & 3 & -1 & -4 & 6 & 0 \\ 1 & -1 & 2 & -1 & 2 & -1 \end{bmatrix}.$$

which, on row reduction, finally leads to

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{11}{2} \\ 0 & 1 & 0 & -\frac{9}{2} & 7 & -\frac{7}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}.$$

Hence,

$$C = (\gamma_{ij}) = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{11}{2} \\ -\frac{9}{2} & 7 & -\frac{7}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}.$$

It is now easy to verify that  $B(\alpha_{ij}) = (\alpha_{ij})C$ .

$$(c) \quad x = \sum_i \alpha_i u_i = 4(0, 1, 1) + 5(1, 0, 1) - (1, 1, 0) = (4, 3, 9)$$

$$y = \sum_i \alpha_i v_i = 4(-1, -2, 1) + 5(1, 3, -1) - (0, -1, 2) = (1, 8, -3).$$

$$\begin{aligned} \text{Now } Ax &= A(4, 3, 9) = A(4(0, 1, 1) + 5(1, 0, 1) - 1(1, 1, 0)) \\ &= 4A(0, 1, 1) + 5A(1, 0, 1) - A(1, 1, 0) \\ &= 4(-1, -1, 1) + 5(1, 3, -1) - (0, -1, 2) \\ &= (1, 8, -3) = y. \end{aligned}$$

$$\begin{aligned} (d) \quad \text{We have } (SA)(0, 1, 1) &= S(A(0, 1, 1)) \\ &= S(-1, -2, 1) = \sum_i \alpha_i v_i \\ &= 1(-1, -2, 1) - 1(0, -1, 2) \\ &= (-1, -1, -1) \end{aligned}$$

$$\begin{aligned} \text{and } (AR)(0, 1, 1) &= A(R(0, 1, 1)) = A(\sum_i \alpha_i u_i) \\ &= A(1(0, 1, 1) - 1(1, 1, 0)) \\ &= A(0, 1, 1) - A(1, 1, 0) \\ &= (-1, -2, 1) - (0, -1, 2) = (-1, -1, -1). \end{aligned}$$

Similarly, we could check that  $SA$  and  $AR$  coincide on the other basis elements  $u_2$  and  $u_3$  also. Hence,  $SA = AR$ .

### Problem Set 7.1

1. Repeat Example 7.1 for the following data :

$$V = V_2$$

$$F = \{(1, -1), (1, 1)\}$$

$$G = \{(2, -3), (3, 3)\}.$$

$$(a) \quad x = (1, 0)$$

$$(b) \quad T: V_2 \rightarrow V_2, T(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$$

$$(c) \quad \text{The given coordinate vector is } (-1, -1)$$

(d) The given matrix is  $\begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$ .

2. Repeat Example 7.1 for the following data :

$$V = V_4$$

$$F = \{(1, 1, 0, 0), (1, 0, 1, 1), (0, 0, 1, 1), (1, 0, -1, 0)\}$$

$G$  = standard basis.

(a)  $x = (1, 1, 1, 1)$

(b)  $T: V_4 \rightarrow V_4, T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + x_3 + x_4, 0, 0, 0)$

(c) The given coordinate vector is  $(1, -2, 3, 0)$

(d) The given matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & -2 & 3 & 4 \end{bmatrix}$$

3. Prove that  $\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$  is similar to  $\begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}$  via the nonsingular matrix

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix}$$

4. Prove that  $\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$  is similar to  $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$  via the non-

singular matrix

$$\begin{bmatrix} 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

5. Prove that  $\begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$  is similar to  $\begin{bmatrix} 6 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$

via the nonsingular matrix

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

6. Prove that similar matrices have the same eigenvalues.

## 7.2 INNER PRODUCT SPACES

In § 2.3 we talked about the orthogonality of vectors in  $V_2$  and  $V_3$ . In this article we shall generalise this concept to all finite-dimensional vector spaces and to certain more general spaces called inner product spaces.

**7.2.1 Definition** Let  $V$  be a (real or complex) vector space. An *inner product* on  $V$  assigns to each ordered pair of vectors  $u, v$  in  $V$  a scalar (complex if  $V$  is a complex vector space and real if  $V$  is a real vector space), written as  $u \cdot v$ , satisfying the following properties:

- (IP1)  $u \cdot (v + w) = u \cdot v + u \cdot w$  for all  $u, v, w$  in  $V$   
 (IP2)  $(\alpha u) \cdot v = \alpha(u \cdot v)$  for all  $u, v \in V$  and all scalars  $\alpha$

(IP3)  $u \cdot v = \overline{v \cdot u}$

(IP4)  $u \cdot u \geq 0$  and  $u \cdot u = 0$  iff  $u = 0$ .

**7.2.2 Definition** A vector space  $V$  together with an inner product is called an *inner product space*.

A finite-dimensional real inner product space is called an *Euclidean space* and a finite-dimensional complex inner product space is called a *unitary space*.

We shall now give some elementary consequences of Definitions 7.2.1 and 7.2.2 in the form of a theorem. Note that we state the results for complex inner product spaces. The corresponding results for real inner product spaces are obtained by merely omitting the complex conjugations.

**7.2.3 Theorem** Let  $V$  be an inner product space,  $u, v$ , and  $w$  be any three vectors in  $V$ , and  $\alpha$  a scalar. Then

- (a)  $(u + v) \cdot w = u \cdot w + v \cdot w$ .  
 (b)  $u \cdot (\alpha v) = \overline{\alpha}(u \cdot v)$ .  
 (c)  $0 \cdot u = 0 = u \cdot 0$ .

*Proof:* (a)  $(u + v) \cdot w = \overline{w \cdot (u + v)} = \overline{w \cdot u + w \cdot v}$   
 $= \overline{w \cdot u} + \overline{w \cdot v} = u \cdot w + v \cdot w.$

$$\begin{aligned} \text{(b)} \quad u \cdot (\alpha v) &= \overline{(\alpha v) \cdot u} = \overline{\alpha(v \cdot u)} \\ &= \overline{\alpha} \overline{(v \cdot u)} = \overline{\alpha} (u \cdot v). \end{aligned}$$

$$\text{(c)} \quad 0 \cdot u = (0v) \cdot u = 0(v \cdot u) = 0.$$

The second part of (c) can be proved similarly. ■

**Example 7.2** In  $V_n^C$  we can define an inner product as follows :

If  $u = (x_1, x_2, \dots, x_n)$  and  $v = (y_1, y_2, \dots, y_n)$ , then

$$u \cdot v = x_1 y_1 + x_2 y_2 + \dots + x_n y_n. \quad (1)$$

For a real vector space  $V_n$ , the analogous inner product is

$$u \cdot v = x_1 y_1 + x_2 y_2 + \dots + x_n y_n. \quad (2)$$

Note that this has already been defined (cf Definition 5.4.2).

It is routine to verify that Equations (1) and (2) are inner products on  $V_n^C$  and  $V_n$ , respectively. Clearly, the familiar dot product on  $V_2$  and  $V_3$  is a special case of the inner product thus defined on  $V_n$ . Therefore,  $V_2$  and  $V_3$ , endowed with the usual dot product, are inner product spaces. This also explains why we use the notation  $u \cdot v$  for the inner product in general inner product spaces.

**7.2.4 Remark** If  $u$  and  $v$  are considered column vectors, then  $u \cdot v$  can also be written as the matrix product  $u^T v$ . On the other hand, if they are considered row vectors, then  $u \cdot v = uv^T$ .

**7.2.5 Remark** The inner product defined in Example 7.2 is called the *natural inner product* on  $V_n^C$  ( $V_n$ ). Problem 5, of Problem Set 7.2, gives another inner product on  $V_n$ .

**Example 7.3** Consider the vector space  $\mathcal{C}[0, 1]$ . Define

$$f \cdot g = \int_0^1 f(t)g(t)dt,$$

where  $f, g \in \mathcal{C}[0, 1]$ . It can be verified that this is an inner product on the space.

Analogously, on  $\mathcal{C}_C[0, 1]$  we can define an inner product

$$f \cdot g = \int_0^1 f(t)\bar{g}(t)dt.$$

Let  $V$  be an inner product space. Then for each vector  $u \in V$ ,  $u \cdot u$  is a nonnegative real number irrespective of whether  $V$  is a real or a complex vector space. We write

$$\|u\| = \sqrt{u \cdot u} \quad (3)$$

and call it the *norm of  $u$* . In Examples 7.2 and 7.3 we have

$$\|(x_1, x_2, \dots, x_n)\| = (|x_1|^2 + |x_2|^2 + \dots + |x_n|^2)^{1/2} \quad (4)$$

$$\text{and} \quad \|f\| = \left( \int_0^1 |f(t)|^2 dt \right)^{1/2}. \quad (5)$$

Equation (4) suggests that the norm of an element  $u$  may be considered the 'distance of  $u$  from the zero of the space' or the 'length of the vector  $u$ '. We shall, without further ado, call  $\|u\|$  the *length of  $u$*  even in the general case. The properties satisfied by  $\|u\|$  in an inner product space are given in the following theorem.

**7.2.6 Theorem** *Let  $V$  be an inner product space. Then for arbitrary vectors  $u$  and  $v$  in  $V$  and scalars  $\alpha$ ,*

- (N1)  $\|\alpha u\| = |\alpha| \|u\|$   
 (N2)  $\|u\| \geq 0$  and  $\|u\| = 0$  iff  $u = 0$   
 (N3)  $|u \cdot v| \leq \|u\| \|v\|$   
 (N4)  $\|u + v\| \leq \|u\| + \|v\|$ .

(N3) is called *Schwarz's inequality* and (N4) is called the *triangle inequality*. The triangle inequality simply generalises the familiar geometrical fact that in a triangle any side is less than the sum of the other two sides.

$$\text{Proof: (N1) } \|\alpha u\|^2 = (\alpha u) \cdot (\alpha u) = (\alpha \bar{\alpha})(u \cdot u) \\ = |\alpha|^2 \|u\|^2,$$

which gives (N1).

(N2) This is a restatement of (IP4).

(N3) This is clearly true if  $u = 0$ . Suppose  $u \neq 0$ . Consider

$$w = v - \alpha u,$$

where  $\alpha = \frac{v \cdot u}{\|u\|^2}$ . Then

$$0 \leq \|w\|^2 = w \cdot w = (v - \alpha u) \cdot (v - \alpha u) \\ = v \cdot v - v \cdot (\alpha u) - (\alpha u) \cdot v + (\alpha u) \cdot (\alpha u) \\ = v \cdot v - (v \cdot u) - \alpha(u \cdot v) + (\alpha \bar{\alpha})(u \cdot u) \\ = \|v\|^2 - \bar{\alpha}(v \cdot u) - \alpha(u \cdot v) + |\alpha|^2 \|u\|^2.$$

Substituting the value of  $\alpha$ , we get

$$0 \leq \|v\|^2 - \frac{2|v \cdot u|^2}{\|u\|^4} + \frac{|v \cdot u|^2}{\|u\|^4} \|u\|^2 \\ = \|v\|^2 - \frac{|v \cdot u|^2}{\|u\|^2},$$

which gives (N3).

$$\begin{aligned} \text{(N4) } \|u + v\|^2 &= (u + v) \cdot (u + v) \\ &= u \cdot u + u \cdot v + \overline{u \cdot v} + v \cdot v \\ &= \|u\|^2 + u \cdot v + \overline{u \cdot v} + \|v\|^2 \\ &= \|u\|^2 + 2 \operatorname{Re}(u \cdot v) + \|v\|^2 \\ &\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2 \\ &\leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2 && \text{(by (N3))} \\ &= (\|u\| + \|v\|)^2, \end{aligned}$$

which gives (N4). ■

Now we are ready to take up the concept of orthogonality of vectors in a general inner product space. Note that these ideas have already been developed for  $V_2$  and  $V_3$  in § 2.3, where we used the dot product without mentioning that it is an inner product on the space concerned.

**7.2.7 Definition** (a) If  $u$  and  $v$  are two vectors in an inner product space  $V$ , they are said to be *orthogonal* if  $u \cdot v = 0$ .

(b) A set of vectors is said to be *orthogonal* if each pair of distinct vectors of the set is orthogonal.

**7.2.8 Theorem** Any orthogonal set of no zero vectors in an inner product space is LI.

*Proof:* Let  $A$  be an orthogonal set and  $B = \{u_1, u_2, \dots, u_n\}$  be a finite subset of  $A$ . Consider

$$\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n = 0.$$

Then, for every  $i = 1, 2, \dots, n$ ,

$$\begin{aligned} 0 &= 0 \cdot u_i \\ &= (\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_n u_n) \cdot u_i \\ &= \alpha_1 (u_1 \cdot u_i) + \alpha_2 (u_2 \cdot u_i) + \dots + \alpha_n (u_n \cdot u_i) \\ &= \alpha_i (u_i \cdot u_i), \text{ since } u_i \cdot u_j = 0 \text{ for } i \neq j. \end{aligned}$$

Since  $u_i \neq 0$ , we get  $\alpha_i = 0$ . Hence, every finite subset of  $A$  is LI and therefore  $A$  is LI. ■

The converse of Theorem 7.2.8 is not true. In other words, not every linearly independent set is orthogonal. This can be seen from simple examples, such as

$$\{2i, 2i + 3j\}$$

in  $V_3$ . But, given a linearly independent set, we can arrive at an orthogonal set in a standard way, which is very important. This construction is called the *Gram-Schmidt orthogonalisation process*. We build this in the proof of Theorem 7.2.10. But first we shall introduce the concept of vector projection, analogous to the situation in  $V_2$  and  $V_3$  (cf Theorem 2.3.8).

**7.2.9 Definition** If  $u$  and  $v$  belong to an inner product space  $V$  and  $v \neq 0$ , then the vector

$$\frac{u \cdot v}{\|v\|^2} v$$

is called the *vector projection of  $u$  along  $v$* .

**7.2.10 Theorem** Every finite-dimensional inner product space has an orthogonal basis.

*Proof:* Let  $\{u_1, u_2, \dots, u_n\}$  be a basis of the inner product space  $V$ . We shall construct an orthogonal set  $\{v_1, v_2, \dots, v_n\}$  of vectors in  $V$ , which is a basis for  $V$ . Write  $v_1 = u_1$ . To construct  $v_2$ , subtract from  $u_2$  its vector projection along  $v_1$ . So



$$v_2 = u_2 - \frac{u_2 \cdot v_1}{\|v_1\|^2} v_1.$$

This gives  $v_2 \cdot v_1 = 0$  and so  $\{v_1, v_2\}$  is an orthogonal set.

Now to construct  $v_3$ , subtract from  $u_3$  its vector projections along  $v_1$  and  $v_2$ . Thus,

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{\|v_1\|^2} v_1 - \frac{u_3 \cdot v_2}{\|v_2\|^2} v_2.$$

This gives  $v_3 \cdot v_1 = 0 = v_3 \cdot v_2$ . (Check.) Hence,  $\{v_1, v_2, v_3\}$  is an orthogonal set. Observe that we have so far used only three of the vectors of the given basis. Proceeding thus and using all the vectors  $u_1, u_2, \dots, u_n$ , we construct the orthogonal set  $B = \{v_1, v_2, \dots, v_n\}$ , where

$$v_i = u_i - \sum_{j=1}^{i-1} \frac{u_i \cdot v_j}{\|v_j\|^2} v_j, \quad i = 2, 3, \dots, n.$$

None of the vectors of  $B$  is zero, because if  $v_k$  were zero, then  $u_k$  would be a linear combination of  $v_1, v_2, \dots, v_{k-1}$  and therefore of  $u_1, u_2, \dots, u_{k-1}$ . By Theorem 7.2.8,  $B$  is a linearly independent set and hence a basis for the  $n$ -dimensional space  $V$ .

**7.2.11 Definition** An orthogonal set of nonzero vectors  $\{u_1, u_2, \dots, u_n\}$  is said to be *orthonormal* if  $\|u_i\| = 1, i = 1, 2, \dots, n$ .

Clearly,  $\{i, j, k\}$  is an orthonormal set (in fact, basis) of  $V_3$ . From Theorem 7.2.10, it follows that every finite-dimensional inner product space has an orthonormal basis.

**7.2.12 Remark** In the Gram-Schmidt construction, described in the proof of Theorem 7.2.10, we started with  $n$  vectors, which formed a basis for  $V$ . Even if we had started with fewer than  $n$  elements, the construction would have worked, provided those elements were LI. The resulting set would have been merely an orthogonal set and not an orthonormal basis.

In Example 7.4 we shall apply the Gram-Schmidt construction to produce an orthonormal set from a given linearly independent set (which is not a basis).

**Example 7.4** Orthonormalise the set of linearly independent vectors  $\{(1, 0, 1, 1), (-1, 0, -1, 1), (0, -1, 1, 1)\}$  of  $V_4$ .

Let  $v_1 = (1, 0, 1, 1)$ . Then

$$\begin{aligned} v_2 &= (-1, 0, -1, 1) - \frac{(-1, 0, -1, 1) \cdot (1, 0, 1, 1)}{3} (1, 0, 1, 1) \\ &= \left(-\frac{4}{3}, 0, -\frac{4}{3}, \frac{2}{3}\right). \end{aligned}$$

$$v_3 = (0, -1, 1, 1) - \frac{(0, -1, 1, 1) \cdot (1, 0, 1, 1)}{3} (1, 0, 1, 1)$$

$$\begin{aligned}
 &= \frac{(0, -1, 1, 1) \cdot (-2/3, 0, -2/3, 4/3)}{24/9} (-2/3, 0, -2/3, 4/3) \\
 &= (-1/2, -1, -1/2, 0).
 \end{aligned}$$

The resulting orthogonal set is

$$\{(1, 0, 1, 1), (-2/3, 0, -2/3, 4/3), (-1/2, -1, -1/2, 0)\}.$$

The corresponding orthonormal set is

$$\{(1/\sqrt{3}, 0, 1/\sqrt{3}, 1/\sqrt{3}), (-1/\sqrt{6}, 0, -1/\sqrt{6}, 2/\sqrt{6}), (-1/\sqrt{3}, -2/\sqrt{3}, -1/\sqrt{3}, 0)\}.$$

**Example 7.5** Find an orthonormal basis of  $\mathcal{G}_3[-1, 1]$  starting from the basis  $\{1, x, x^2, x^3\}$ . Use the inner product defined by

$$f \cdot g = \int_{-1}^1 f(t)g(t)dt.$$

We take  $v_1 = 1$ .

$$\begin{aligned}
 \text{Then } v_2 &= x - \frac{x \cdot 1}{2} 1 \\
 &= x - \left(\frac{1}{2} \int_{-1}^1 t dt\right) = x. \\
 v_3 &= x^2 - \frac{x^2 \cdot 1}{2} 1 - \frac{x^2 \cdot x}{2/3} x \\
 &= x^2 - \frac{1}{2} \int_{-1}^1 t^2 dt - \frac{2}{3} x \int_{-1}^1 t^3 dt = x^2 - \frac{1}{2}. \\
 v_4 &= x^3 - \frac{x^3 \cdot 1}{2} 1 - \frac{x^3 \cdot x}{2/3} x - \frac{x^3 \cdot (x^2 - 1/2)}{2/5} (x^2 - \frac{1}{2}) \\
 &= x^3 - \frac{3}{8}x.
 \end{aligned}$$

Thus, the orthogonal basis is

$$\{1, x, x^2 - \frac{1}{2}, x^3 - \frac{3}{8}x\}.$$

To get the corresponding orthonormal basis, we divide these by the respective norms and get

$$\left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}x, \frac{3\sqrt{5}}{2\sqrt{2}}(x^2 - \frac{1}{2}), \frac{5\sqrt{7}}{2\sqrt{2}}(x^3 - \frac{3}{8}x) \right\}.$$

### Problem Set 7.2

1. Use the Gram-Schmidt process to orthonormalise the linearly independent sets of vectors given in Problem 1, Problem Set 3.5.
2. Repeat Problem 1 for Problem 2, Problem Set 3.5.
3. Repeat Problem 1 for Problem 3, Problem Set 3.5.

(Take the inner product in the space to be  $\int_{-1}^1 f(t)g(t)dt$ .)

4. In an inner product space  $V$  show that

(a) If  $v \cdot u = 0$  for all  $u \in V$ , then  $v = 0$

(b) If  $v \cdot u = w \cdot u$  for all  $u \in V$ , then  $v = w$ .

5. Show that an inner product can be defined on  $V_2$  by

$$(x_1, x_2) \cdot (y_1, y_2) = \frac{(x_1 - x_2)(y_1 - y_2)}{4} + \frac{(x_1 + x_2)(y_1 + y_2)}{4}.$$

In this inner product space calculate

(a)  $e_1 \cdot e_2$       (b)  $(1, -1) \cdot (1, 1)$ .

6. (This is a generalisation of Problem 5.) Let  $B$  be a basis of an  $n$ -dimensional vector space  $V$ . Define, for arbitrary  $x, y \in V$ ,

$$x \cdot y = [x]_B \cdot [y]_B,$$

where the inner product of coordinate vectors on the right-hand side is the natural inner product of  $V_n^C$ . Prove that this definition gives an inner product on  $V$ . In this inner product space prove that the basis  $B$  is orthonormal.

7. Prove that all the eigenvalues of a symmetric matrix are real.  
8. Let the set  $\{v_1, v_2, \dots, v_n\}$  be LD. What happens when the Gram-Schmidt process of orthogonalisation is applied to it?

### 7.3 ORTHOGONAL AND UNITARY MATRICES

We shall discuss two special types of matrices in this article, and assume the natural inner product on  $V_n$  and  $V_n^C$ .

**7.3.1 Definition** (a) A real square matrix  $H$  is called an *orthogonal matrix* if

$$H^T H = I = H H^T.$$

In other words,  $H^T = H^{-1}$ .

(b) A complex square matrix  $U$  is called a *unitary matrix* if

$$U^* U = I = U U^*.$$

In other words,  $U^* = U^{-1}$ .

For example,

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \text{ is orthogonal and } \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ is unitary.}$$

Orthogonal and unitary matrices are important because they correspond to certain linear operators that have significant properties. But, since we shall not be discussing such operators in the general setting of a vector space, we shall be content with deriving certain fundamental properties of orthogonal and unitary matrices.

**7.3.2 Theorem** A real (complex) square matrix is orthogonal (unitary) iff the rows of the matrix form an orthonormal set of vectors or iff the columns of the matrix form an orthonormal set of vectors.

**Proof:** We shall prove the theorem for the complex case. By removing all the complex conjugations in the proof, we get the proof for the real case.

Let  $u_1, u_2, \dots, u_n$  be the column vectors of a complex square matrix  $U$ . Then

$$U = [u_1 \ u_2 \ \dots \ u_n] \text{ and } U^* = \begin{bmatrix} \bar{u}_1^T \\ \bar{u}_2^T \\ \vdots \\ \bar{u}_n^T \end{bmatrix},$$

where  $u_i = (u_{i1}, u_{i2}, u_{in})^T$  and  $\bar{u}_i^T = (\bar{u}_{i1}, \bar{u}_{i2}, \dots, \bar{u}_{in})$ .

$$\begin{aligned} \text{So } U^*U &= \begin{bmatrix} \bar{u}_1^T \\ \bar{u}_2^T \\ \vdots \\ \bar{u}_n^T \end{bmatrix} [u_1 \ u_2 \ \dots \ u_n] \\ &= \begin{bmatrix} \bar{u}_1^T u_1 & \bar{u}_1^T u_2 & \dots & \bar{u}_1^T u_n \\ \bar{u}_2^T u_1 & \bar{u}_2^T u_2 & \dots & \bar{u}_2^T u_n \\ \vdots & \vdots & & \vdots \\ \bar{u}_n^T u_1 & \bar{u}_n^T u_2 & \dots & \bar{u}_n^T u_n \end{bmatrix}. \end{aligned}$$

This is the identity matrix  $I$  iff

$$\bar{u}_i^T u_j = 0 \quad \text{for } i \neq j$$

and

$$\bar{u}_i^T u_i = 1 \quad \text{for all } i = 1, 2, \dots, n.$$

This means that the set of column vectors  $\{u_1, u_2, \dots, u_n\}$  is orthonormal.

To prove that this requirement is equivalent to saying that the rows are orthonormal, we have to note only that  $U^*U = I$  implies  $UU^* = I$  (and vice versa), and this latter is true iff the columns of  $U^*$  are orthonormal. ■

**7.3.3 Theorem** (a) *The orthogonal  $n \times n$  matrices form a group for multiplication.*

(b) *The unitary  $n \times n$  matrices form a group for multiplication.*

The proof is left to the reader. The groups mentioned in the theorem are called the *orthogonal group* of order  $n$  and the *unitary group* of order  $n$ , respectively.

We shall end this article by giving a definition.

**7.3.4 Definition** (a) Two square matrices  $A$  and  $B$  of the same order

are said to be *orthogonally similar* if  $A = HBH^{-1}$  for some orthogonal matrix  $H$ .

(b) Two square matrices  $A$  and  $B$  of the same order are said to be *unitarily similar* if  $A = UBU^{-1}$  for some unitary matrix  $U$ .

### Problem Set 7.3

1. If  $H$  is orthogonal, prove that  $\det H = \pm 1$ .
2. If  $U$  is unitary, prove that  $|\det U| = 1$ .
3. If  $U$  is unitary, show that  $\bar{U}$ ,  $U^T$ , and  $U^k$  ( $k$  a positive integer) are also unitary.
4. If  $H$  is orthogonal, show that  $H^T$  and  $H^k$  ( $k$  a positive integer) are also orthogonal.
5. Show that

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

are unitary, involutory, as well as Hermitian.

6. Prove that if  $U$  is unitary and  $U^*AU$  and  $U^*BU$  are both diagonal matrices, then  $AB = BA$ . Is this result true if  $U$  is replaced by a real orthogonal matrix  $H$ ?
7. Let  $T$  be the linear operator on  $V_3$  defined by  $T(x, y, z) = (x', y', z')$ , where  $x' = x \cos \phi - y \sin \phi$ ,  $y' = x \sin \phi + y \cos \phi$ ,  $z' = z$ , with respect to a cartesian coordinate system. Prove that  $T$  is given by an orthogonal matrix.
8. True or false?
  - (a) Every real symmetric matrix is orthogonal.
  - (b) Every Hermitian matrix is unitary.
  - (c) If for a complex matrix  $A$ ,  $AA^* = A^*A$ , then  $A$  is unitary.
  - (d) The eigenvalues of a unitary matrix have absolute value 1.
  - (e) If a matrix is unitary, it is also Hermitian.
  - (f) The columns of

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

form an orthonormal set of vectors.

## 7.4 APPLICATION TO REDUCTION OF QUADRICS

In this article we shall use the concept of an orthogonal matrix to reduce a real symmetric matrix to the diagonal form by a similarity

transformation. An analogous reduction of Hermitian matrices to diagonal form by a similarity transformation using unitary matrices is possible; such a reduction and related topics are covered in advanced textbooks. Since the problem of reduction of a real symmetric matrix to diagonal form is the same as the geometrical problem of reduction of a quadric to its principal axes, we start this article with a discussion of the geometrical aspect.

In 2-dimensional analytic geometry the equation of a central conic referred to its centre as origin is of the form

$$ax^2 + 2hxy + by^2 = k.$$

This can be written in matrix form as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k.$$

In 3-dimensional analytic geometry a similar equation represents what is called a central *quadric*. It is of the form

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = k.$$

In detail, this equation is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = k.$$

In general, even in  $n$  dimensions, we can write a similar equation. The surface represented by it is called a quadric in  $n$  dimensions or simply a quadric. Its equation is of the form

$$u^T A u = k,$$

where

$$u = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and  $A$  is a real symmetric matrix of order  $n$ .

In discussing a quadric in  $V_n$  we shall use geometrical ideas like tangent and normal. A tangent to a quadric is a line that meets the quadric at two or more coincident points. If a straight line is perpendicular to the set of all tangents at a point, we say it is a normal at the point. We assume the natural inner product on  $V_n$  and begin with a fundamental lemma.

**7.4.1 Lemma** If  $u_0$  is a point on the quadric  $u^T A u = k$ , then the normal at  $u_0$  is in the direction of  $A u_0$ .

**Proof:** Figure 7.1 represents the situation in the 2-dimensional case. Let  $L = \{u_0 + tv \mid t \in \mathbb{R}\}$  be an arbitrary tangent, in the direction of  $v$ , at  $u_0$  on the surface.  $L$  clearly passes through the point  $u_0$ . Any other point, say  $u_0 + tv$ , where this line  $L$  meets the quadric, will be given by those values of  $t$  that are roots of  $(u_0 + tv)^T A (u_0 + tv) = k$ . This is a scalar equation, though written in matrix form. It simplifies to

$$u_0^T A u_0 + tv^T A u_0 + u_0^T A(tv) + t^2 v^T A v = k.$$

But  $u_0^T A u_0 = k$ , since  $u_0$  lies on the quadric. So the equation becomes

$$t(v^T A u_0 + u_0^T A v) + t^2 v^T A v = 0.$$

One root of this is  $t = 0$ , which corresponds to the point  $u_0$  itself. In order that  $L$  may be a tangent to the surface, we require the other root also to be zero. This gives

$$v^T A u_0 + u_0^T A v = 0.$$

But  $v^T A u_0$  and  $u_0^T A v$  represent the same real scalar. Hence,

$$v^T A u_0 = 0 = u_0^T A v.$$

The first equation means  $Au_0$  is perpendicular to  $v$ , i.e. to  $u_0 + tv$ , an arbitrary tangent at  $u_0$ . Thus,  $Au_0$  is perpendicular to all tangents at  $u_0$ . In other words,  $Au_0$  gives the direction of the normal at  $u_0$ . ■

Let  $u_0$  be a point on a quadric. Then the vector  $u_0$  is said to determine the direction of a principal axis if it is normal to the quadric at the point  $u_0$ . By Lemma 7.4.1, it follows that the principal axes are in the directions  $u_0$ , where  $Au_0 = \lambda u_0$  for some real scalars  $\lambda$ . Thus, we have established the following theorem.

**7.4.2 Theorem** *In the quadric  $u^T A u = k$  the principal axes of the quadric are in the direction of the eigenvectors of  $A$ .*

**Example 7.6** Find the directions of the principal axes of the conic  $10x^2 + 4xy + 7y^2 = 100$ .

In matrix form this is written as

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 100.$$

The eigenvalues of  $A = \begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$  are given by

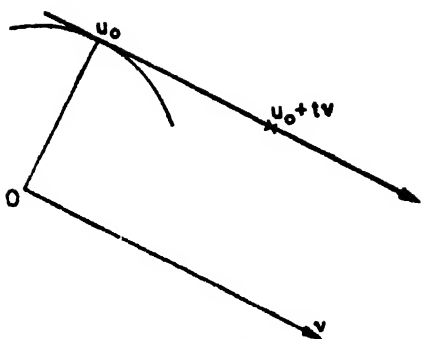


FIGURE 7.1

$$\begin{vmatrix} 10 - \lambda & 2 \\ 2 & 7 - \lambda \end{vmatrix} = 0,$$

which gives  $\lambda = 11, 6$ .

Corresponding to the eigenvalue  $\lambda = 11$ , the eigenvectors are given by

$$\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives an eigenvector  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

Again, corresponding to the eigenvalue  $\lambda = 6$ , the eigenvectors are given by

$$\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This gives an eigenvector  $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ . Therefore, the axes of the conic are along the directions  $(2i + j)$  and  $(i - 2j)$ .

In Example 7.6 we have determined two eigenvectors corresponding to two distinct eigenvalues. Naturally they are orthogonal. Let us normalise these vectors and construct the matrix

$$\begin{bmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{bmatrix} = H \text{ (say).}$$

Observe that this is an orthogonal matrix (since its columns are orthonormal). Use this as a similarity transformation on the given symmetric matrix

$$A = \begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}.$$

Calculate  $H^T A H$ . We obtain the matrix

$$D = \begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}.$$

Note that this matrix is a diagonal matrix whose diagonal elements are the eigenvalues of  $A$ . Thus,  $A = \begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$  is orthogonally similar to the diagonal matrix  $D = \begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}$ . The passage from  $A$  to  $D$  is called



**diagonalisation.** The geometrical implication of this diagonalisation can be understood by going back to the equation

$$10x^2 + 4xy + 7y^2 = 100.$$

This equation contains the  $xy$  term; so we do not know the exact size and location of the conic except that it is an ellipse, since its discriminant  $h^2 - ab = -66 < 0$ . To find the length of the axes we transform the

coordinates  $\begin{bmatrix} x \\ y \end{bmatrix}$  to the new coordinates  $\begin{bmatrix} x' \\ y' \end{bmatrix}$  by the formulae of transformation

$$\begin{aligned} x &= x' \cos \theta + y' \sin \theta \\ y &= -x' \sin \theta + y' \cos \theta, \end{aligned}$$

where  $\theta$  is the angle of rotation of the axes. We know from analytic geometry that if  $\theta$  is suitably chosen, the  $xy$  term will vanish in the new coordinates, and the lengths of the axes can be found by inspection of the transformed equation. The theorem on the reduction of a real symmetric matrix, which we shall now prove, will produce this value of  $\theta$  without the help of geometry. This theorem of linear algebra says that the required

transformation matrix from  $\begin{bmatrix} x \\ y \end{bmatrix}$  to  $\begin{bmatrix} x' \\ y' \end{bmatrix}$  is nothing but the orthogonal matrix obtained from the normalised eigenvectors of  $A$ . In Example 7.6 it is just

$$H = \begin{bmatrix} 2 & 1 \\ \sqrt{5} & \sqrt{5} \\ 1 & -2 \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}.$$

Using this, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ \sqrt{5} & \sqrt{5} \\ 1 & -2 \\ \sqrt{5} & -\sqrt{5} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}.$$

Write this as  $u = Hv$ . Then  $u^T A u = k$  becomes  $v^T H^T A H v = k$ . This simplifies to

$$[x' \ y'] \begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 100,$$

i.e.

$$11x'^2 + 6y'^2 = 100$$

or

$$\frac{x'^2}{(10/\sqrt{11})^2} + \frac{y'^2}{(10/\sqrt{6})^2} = 1.$$

This is an ellipse with lengths of semiaxes  $10/\sqrt{11}$  and  $10/\sqrt{6}$ . This analysis shows that

(i) the eigenvectors of  $A$  give the directions of the principal axes of the conic;

(ii) the eigenvalues  $\lambda_1$  and  $\lambda_2$  of  $A$  specify the lengths of the semiaxes  $\sqrt{k/\lambda_1}$  and  $\sqrt{k/\lambda_2}$ ;

(iii) the orthogonal matrix  $H$  constructed out of the normalised eigenvectors of  $A$  is the matrix that reduces  $A$  to the diagonal form; and

(iv) the matrix  $A$  is orthogonally similar to a diagonal matrix, whose diagonal elements are the eigenvalues of  $A$ .

We shall now prove that this situation is true in the  $n$ -dimensional case also, at least for the case where the matrix has distinct eigenvalues. The result of the theorem is true even for general real symmetric matrices. But we leave the theoretical development of this case to more advanced textbooks.

**7.4.3 Theorem** (Diagonalisation of a real symmetric matrix) *Let  $A$  be a real symmetric matrix with distinct eigenvalues. Let the normalised eigenvectors of  $A$  be written as column vectors of a matrix  $H$ . Then*

(a)  $H$  is orthogonal.

(b)  $H^T A H (= D)$  is a diagonal matrix, whose diagonal entries are the eigenvalues of  $A$ .

(c)  $A$  is orthogonally similar to the diagonal matrix  $D$ .

*Proof:* (a) First, we prove that the eigenvectors of  $A$  corresponding to distinct eigenvalues are orthogonal. Let  $u_1$  and  $u_2$  be eigenvectors corresponding to distinct eigenvalues  $\lambda_1$  and  $\lambda_2$ , respectively. Note that both are real (see Problem 7, Problem Set 7.2). Then

$$\begin{aligned} (\lambda_1 - \lambda_2)(u_1 \cdot u_2) &= \lambda_1(u_1 \cdot u_2) - \lambda_2(u_1 \cdot u_2) \\ &= (\lambda_1 u_1 \cdot u_2) - (u_1 \cdot \lambda_2 u_2) \\ &= (A u_1 \cdot u_2) - (u_1 \cdot A u_2) \\ &= (A u_1)^T u_2 - u_1^T (A u_2) \\ &= u_1^T A^T u_2 - u_1^T A u_2 \\ &= 0 \end{aligned} \quad (\text{because } A^T = A).$$

Since  $\lambda_1 \neq \lambda_2$ , it follows that  $u_1 \cdot u_2 = 0$ . This coupled with Theorem 7.3.2 shows that  $H$  is an orthogonal matrix.

$$(b) \quad H = [u_1 \ u_2 \ \dots \ u_n],$$

where  $u_i$  are the eigenvectors of  $A$  written as column vectors.

$$\text{So} \quad H^T A = \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} A = \begin{bmatrix} u_1^T A \\ u_2^T A \\ \vdots \\ u_n^T A \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} (Au_1)^T \\ (Au_2)^T \\ \vdots \\ (Au_n)^T \end{bmatrix} = \begin{bmatrix} (\lambda_1 u_1)^T \\ (\lambda_2 u_2)^T \\ \vdots \\ (\lambda_n u_n)^T \end{bmatrix} \quad (\text{where the } \lambda_i \text{'s are the corresponding eigenvalues}) \\
 &= \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix} \\
 &\quad \Rightarrow DH^T = DH^{-1}.
 \end{aligned}$$

Hence,  $H^T A H = D$  as required.

(c) This is only a restatement of (b).  $\square$

**Example 7.7** Reduce the quadric

$$[x \ y \ z] \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 36 \quad (1)$$

to its principal axes. Note that when this equation is written in full it takes the form

$$7x^2 + 7y^2 - 2z^2 + 20yz - 20zx - 2xy = 36.$$

According to the preceding discussion we need to reduce the matrix

$$A = \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix}$$

to the diagonal form. The eigenvalues of  $A$  are 6,  $-12$ , and  $18$ . The corresponding eigenvectors are  $(1, 1, 0)$ ,  $(1, -1, 2)$ , and  $(1, -1, -1)$ . Hence, the normalised eigenvectors are

$(1/\sqrt{2}, 1/\sqrt{2}, 0)$ ,  $(1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6})$ , and  $(1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3})$ .

These three vectors give the following orthogonal matrix

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

Hence,  $H^T A H$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 7 & -1 & -10 \\ -1 & 7 & 10 \\ -10 & 10 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

is the required diagonal matrix.

So the equation of the quadric referred to its axes is

$$6x^2 - 12y^2 + 18z^2 = 36,$$

i.e. 
$$\frac{x^2}{6} - \frac{y^2}{3} + \frac{z^2}{2} = 1.$$

This is a hyperboloid of one sheet.

We shall conclude this article with another example where the matrix does not possess distinct eigenvalues. Though we have not developed the necessary theory for this, the following example will show that a reduction to diagonal form is possible even in such a case.

**Example 7.8** Reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$$

to the diagonal form, and hence reduce the quadric

$$x^2 + y^2 + z^2 + 4yz - 4zx + 4xy = 27 \quad (2)$$

to its principal axes.

The eigenvalues of the matrix  $A$  are 3, 3, and  $-3$ . Note that in this case 3 is a repeated eigenvalue. So Theorem 7.4.3, which depends on the fact that the matrix has distinct eigenvalues, cannot be applied. However, an orthogonal matrix can be obtained by looking at the eigenvectors corresponding to 3 and  $-3$ .

The eigenvectors corresponding to the eigenvalue 3 are given by

$$\begin{bmatrix} -2 & 2 & -2 \\ 2 & -2 & 2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives only one equation :

$$x - y + z = 0;$$

The solution set of this is the subspace  $\{(y - z, y, z) \mid y, z \in R\}$  of  $V_3$ . We can choose two linearly independent vectors in this 2-dimensional space by giving suitable values to  $y$  and  $z$ . Taking  $y = 1$  and  $z = 1$ , we get  $(0, 1, 1)$ ; and taking  $y = 1$  and  $z = 2$ , we get  $(-1, 1, 2)$ . Since these two are LI, we have only to orthogonalise them. Using the Gram-Schmidt process, we get two orthogonal vectors

$$(0, 1, 1) \text{ and } (-1, -1/2, 1/2).$$

Normalising them, we get

$$(0, 1/\sqrt{2}, 1/\sqrt{2}) \text{ and } (-\sqrt{2}/\sqrt{3}, -1/\sqrt{6}, 1/\sqrt{6}).$$

The third eigenvector is the one that corresponds to the eigenvalue  $-3$  and is  $(1, -1, 1)$ .

Normalising this vector, we get  $(1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3})$ . Without checking, we can say that this will be orthogonal to the two eigenvectors corresponding to 3, because eigenvectors corresponding to distinct eigenvalues are orthogonal. Thus, we have the orthogonal matrix

$$H = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Hence,  $H^T A H$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \\ = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

Thus, the quadric (2) reduces to

$$3x^2 + 3y^2 - 3z^2 = 27,$$

i.e.

$$\frac{x^2}{9} + \frac{y^2}{9} - \frac{z^2}{9} = 1.$$

It may be noted here that Equation (2) can be written as

$$[x \ y \ z] \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 27.$$

### Problem Set 7.4

1. Reduce the following matrices to diagonal form :

$$(a) \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix} \quad (b) \begin{bmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{bmatrix} \quad (c) \begin{bmatrix} 4 & 3 & 3 \\ 3 & 0 & -1 \\ 3 & -1 & 3 \end{bmatrix}.$$

2. Reduce the following conics to their principal axes :

$$\begin{aligned} (a) \quad & 7x^2 + 52xy - 32y^2 = 180 \\ (b) \quad & 17x^2 + 312xy + 108y^2 = 900 \\ (c) \quad & 145x^2 + 120xy + 180y^2 = 900. \end{aligned}$$

## Appendix

# Ordinary Linear Differential Equations

In § 4.9 we saw that the general solution of the  $n$ -th order normal linear differential equation

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x)y = g(x) \quad (1)$$

is of the form  $y = y_c + y_p$ , where  $y_c$  is the solution of the associated homogeneous equation

$$a_0(x) \frac{d^n y}{dx^n} + a_1(x) \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n(x)y = 0,^* \quad (2)$$

and  $y_p$  is one particular solution of (1).  $y_c$  is called the complementary function and is the kernel of the linear differential operator

$$L = a_0(x) \frac{d^n}{dx^n} + a_1(x) \frac{d^{n-1}}{dx^{n-1}} + \dots + a_n(x). \quad (3)$$

Thus, the method of solving Equation (1) involves two steps, namely, finding  $y_c$  and  $y_p$ . Further, it may be noted that the solution space of Equation (2) is  $n$ -dimensional (cf Theorem 4.9.3).

The solution of the first order normal linear differential equation has thus been completely discussed in § 4.9. In this appendix we shall develop methods of finding  $y_c$  and  $y_p$  for differential equations of arbitrary order.

## AI HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

In this article we shall discuss the method of finding  $y_c$  for linear differential equations with constant coefficients, i.e. the method of solving

\* The zero on the right-hand side of Equation (2) should be 0. But throughout the appendix we shall use 0 as is the practice in writing a differential equation. The context will make it clear whether it is 0 or 0.

homogeneous linear differential equations with constant coefficients. To start with, let us consider the second order differential equation

$$a_0 y'' + a_1 y' + a_2 y = 0, \quad (4)$$

where  $a_0$ ,  $a_1$ , and  $a_2$  are constants, and  $a_0 \neq 0$ . This equation may also be written as

$$(a_0 D^2 + a_1 D + a_2)y = 0 \quad (5)$$

or

$$Ly = 0, \quad (6)$$

where  $L = a_0 D^2 + a_1 D + a_2$  is a linear operator from  $\mathcal{C}^{(2)}(I)$  to  $\mathcal{C}(I)$ .

It may be noted that  $L$  is a linear operator also from  $\mathcal{C}_C^{(2)}(I)$  to  $\mathcal{C}_C(I)$ .

In § 4.9 we saw that the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has the complementary function  $y_0 = Ce^{\int P(x)dx}$ . When  $P(x)$  is a constant  $p$ , this reduces to  $y_0 = Ce^{px}$ . Thus, the first order homogeneous linear differential equation

$$\frac{dy}{dx} + ay = 0$$

has the solution  $Ce^{ax}$ . Using this as an analogy, we try  $e^{mx}$  (where  $m$  is a constant) as a solution of the second order homogeneous linear differential equation

$$(a_0 D^2 + a_1 D + a_2)y = 0.$$

This gives

$$(a_0 m^2 + a_1 m + a_2)e^{mx} = 0.$$

But  $e^{mx}$  is never zero. So

$$a_0 m^2 + a_1 m + a_2 = 0. \quad (7)$$

Equation (7) is called the *indicial equation* or the *auxiliary equation*. The values of  $m$  for which Equation (7) holds will lead to solutions  $e^{mx}$  of the differential equation (5). Let the roots of Equation (7) be  $m_1$  and  $m_2$ . Then  $e^{m_1 x}$  and  $e^{m_2 x}$  are two solutions. Their Wronskian is

$$W[e^{m_1 x}, e^{m_2 x}] = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = (m_2 - m_1)e^{(m_1 + m_2)x}.$$

The different roots of the auxiliary equation (7) give rise to the following three cases.

**Case 1**  $m_1 \neq m_2$  In this case the Wronskian is never zero (cf Theorem 4.9.2). So  $e^{m_1 x}$  and  $e^{m_2 x}$  are two linearly independent solutions of Equation (5). The solution space has dimension 2. Therefore,  $e^{m_1 x}$  and  $e^{m_2 x}$  form a basis for the solution space. Hence, the general solution is

$$y_0 = C_1 e^{m_1 x} + C_2 e^{m_2 x}.$$



**Case 2**  $m_1 = m_2$ . In this case it follows from Equation (7) that  $2m_1 = -a_1/a_0$  and one solution of Equation (5) is  $y_1 = e^{m_1 x}$ ,  $m_1 = -a_1/2a_0$ . Let  $y_2$  be the other solution such that  $y_1$  and  $y_2$  are LI. Then

$$W[e^{m_1 x}, y_2] = Ce^{-\int \frac{a_1}{a_0} dx} = Ce^{2m_1 x},$$

by Abel's formula (cf Remark 6.9.4). This, on simplification, gives

$$e^{m_1 x} y_2' - m_1 y_2 e^{m_1 x} = Ce^{2m_1 x}.$$

Hence,

$$y_2' - m_1 y_2 = Ce^{m_1 x}.$$

Solving this first order linear differential equation, we obtain a particular solution  $y_2 = Cxe^{m_1 x}$ . This gives us  $y_2 = xe^{m_1 x}$  as a second solution (taking  $C = 1$ ) of Equation (5). Obviously,  $y_1 = e^{m_1 x}$  and  $y_2 = xe^{m_1 x}$  are LI. (Why?) Hence, the general solution is

$$y_c = C_1 e^{m_1 x} + C_2 x e^{m_1 x}.$$

Thus, we have obtained  $y_c$ .

**Example A.1** Obtain  $y_c$ , the complementary function for the differential equation

$$(D^2 - 2D - 1)y = \sin x.$$

The auxiliary equation of the associated homogeneous equation is

$$m^2 - 2m - 1 = 0.$$

Its roots are  $m_1 = 1 + \sqrt{2}$ ,  $m_2 = 1 - \sqrt{2}$ . So  $e^{(1+\sqrt{2})x}$  and  $e^{(1-\sqrt{2})x}$  are two linearly independent solutions of the associated homogeneous equation. Hence,

$$y_0 = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}.$$

**Example A.2** Consider the differential equation

$$(4D^2 + 12D + 9)y = xe^{-3x/2}.$$

The auxiliary equation of the associated homogeneous equation is

$$4m^2 + 12m + 9 = 0.$$

Its roots are  $m_1 = -3/2 = m_2$ . As the roots are equal, the two linearly independent solutions are  $e^{-3x/2}$  and  $xe^{-3x/2}$ . Thus, the complementary function is

$$y_0 = C_1 e^{-3x/2} + C_2 x e^{-3x/2}.$$

**Example A.3** Consider the differential equation

$$(D^2 - 4D + 13)y = x^2 + \cos 2x.$$

The auxiliary equation of the associated homogeneous equation is

$$m^2 - 4m + 13 = 0.$$

Its roots are  $m_1 = 2 + 3i$ ,  $m_2 = 2 - 3i$ . Since  $m_1 \neq m_2$ , two linearly independent solutions are  $e^{(2+3i)x}$  and  $e^{(2-3i)x}$ . Thus, the complementary function is

$$y_0 = C_1 e^{(2+3i)x} + C_2 e^{(2-3i)x}.$$

This is a complex solution. We shall now discuss the method of getting real solutions from complex solutions.

**A1.1 Lemma** If  $u(x) + iv(x)$  is a complex solution of a homogeneous linear differential equation  $Ly = 0$  with real coefficients, then both  $u(x)$  and  $v(x)$  separately satisfy the equation  $Ly = 0$ .

*Proof:* Since  $u(x) + iv(x)$  is a solution of  $Ly = 0$ , we have

$$L(u(x) + iv(x)) = 0.$$

$L$  is a linear operator. Therefore,

$$L(u(x)) + iL(v(x)) = 0.$$

Equating the real and imaginary parts, we get

$$L(u(x)) = 0 \text{ and } L(v(x)) = 0.$$

Hence the lemma. ■

**Case 3 Auxillary equation with complex roots** If  $\alpha + i\beta$  is a root of the auxillary equation (with real coefficients), then  $\alpha - i\beta$  is another root. Since  $m_1 \neq m_2$ , two linearly independent solutions are  $e^{(\alpha+i\beta)x}$  and  $e^{(\alpha-i\beta)x}$ . These are complex solutions. As

$$e^{(\alpha+i\beta)x} = e^{\alpha x} \cos \beta x + ie^{\alpha x} \sin \beta x$$

is a solution of the differential equation (5), it follows from Lemma A1.1 that  $e^{\alpha x} \cos \beta x$  and  $e^{\alpha x} \sin \beta x$  are two real solutions of Equation (5). Further, they are linearly independent, because

$$W[e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x] = \beta e^{\alpha x}$$

is never zero. Hence, the complementary function can also be written as

$$y_c = C_1 e^{\alpha x} \cos \beta x + C_2 e^{\alpha x} \sin \beta x.$$

If  $C_1$  and  $C_2$  are taken to be real, then we get all real solutions of Equation (5). For instance, in Example A.3 the real complementary function is

$$y_c = e^{2x}(C_1 \cos 3x + C_2 \sin 3x).$$

We now state (without proof) the extension of the aforesaid method of finding  $y_c$  for differential equations of arbitrary order. Consider the  $n$ -th order linear differential equation

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = 0, \quad (8)$$

where  $a_0, a_1, \dots, a_n$  are real constants. In this case also we try  $y = e^{mx}$  as a solution of Equation (8) and obtain the auxillary equation

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0. \quad (9)$$

This has  $n$  roots (which may be complex). Some of the roots may be repeated.

For each real root  $\alpha$  with multiplicity  $r$  (i.e. repeated  $r$  times) of the

auxiliary equation, the corresponding part of the complementary function is

$$e^{\alpha x}(A_0 + A_1x + \dots + A_{r-1}x^{r-1}),$$

where  $A_0, A_1, \dots, A_{r-1}$  are arbitrary constants.

For each pair of complex roots  $\alpha \pm i\beta$  with multiplicity  $r$ , the corresponding part of the complementary function is

$$e^{\alpha x}(A_0 + A_1x + \dots + A_{r-1}x^{r-1}) \cos \beta x \\ + e^{\alpha x}(B_0 + B_1x + \dots + B_{r-1}x^{r-1}) \sin \beta x.$$

Let us illustrate these rules through a few examples.

**Example A.4** Find the complementary function of the differential equation

$$y''' + y' = 2x^2 + 4 \sin x.$$

The auxiliary equation is  $m^3 + m = 0$ . Its roots are  $m = 0, i, -i$ . So the complementary function is

$$y_c = C_1 + C_2 \cos x + C_3 \sin x.$$

**Example A.5** Consider the differential equation

$$(D^6 - 10D^5 + 43D^4 - 100D^3 + 131D^2 - 90D + 25)y = x^2.$$

The auxiliary equation is

$$m^6 - 10m^5 + 43m^4 - 100m^3 + 131m^2 - 90m + 25 = 0$$

or

$$(m - 1)^2(m^2 - 4m + 5)^2 = 0.$$

Its roots are  $m_1 = m_2 = 1$ ;  $m_3 = m_4 = 2 + i$ ;  $m_5 = m_6 = 2 - i$ .

Corresponding to the repeated root '1', the complementary function is

$$e^x(C_1 + C_2x),$$

and corresponding to the repeated complex roots, the complementary function is

$$e^{2x}\{(C_3 + C_4x) \cos x + (C_5 + C_6x) \sin x\}.$$

Hence, the complementary function is

$$y_c = e^x(C_1 + C_2x) + e^{2x}\{(C_3 + C_4x) \cos x + (C_5 + C_6x) \sin x\}.$$

### Problem Set A1

1. Solve the following homogeneous linear differential equations :

(a)  $y'' + 4y' + 4y = 0$

(b)  $y'' + 4y = 0$

(c)  $y''' + y' = 0$

(d)  $3y'' + y' - 2y = 0$

(e)  $y''' - y = 0$

(f)  $y'' - 2y' - 3y = 0$

(g)  $y'' + 2y' - y = 0$

(h)  $y''' - y'' - y' + y = 0$

(i)  $(D^4 + 4D^3 - 5D^2)y = 0$

(j)  $(D^3 - D^2 - 6D)y = 0$

(k)  $(D^3 + 4D + 11)^2y = 0$

(l)  $(D^2 - 1)^2y = 0$

(m)  $(D^3 - 3D + 2)^2y = 0$

(n)  $(D^3 + 3D + 1)y = 0$

(o)  $(2D^3 + 5D^2 - 3D)y = 0$

(p)  $(D^3 - 4D^2 + 4D)y = 0.$

## A2 METHOD OF VARIATION OF PARAMETERS

In this and the next two articles we shall give various methods of finding  $y_p$  for a nonhomogeneous ordinary linear differential equation. One method universally applicable to all linear equations (even to some nonlinear equations) is the method of *variation of parameters*.

We shall illustrate the method by taking a second order equation

$$(a_0 D^2 + a_1 D + a_2)y = g(x), \quad a_0 \neq 0. \quad (1)$$

Let the complementary function  $y_c$  be

$$y_c = C_1 y_1(x) + C_2 y_2(x),$$

where  $C_1$  and  $C_2$  are arbitrary constants, and  $y_1(x)$  and  $y_2(x)$  are two linearly independent solutions of the associated homogeneous equation of Equation (1). In this method we assume the particular solution  $y_p$  to be of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x), \quad (2)$$

where  $u_1(x)$  and  $u_2(x)$ , yet to be chosen, are twice differential functions of  $x$ . Our assumption will be justified if we can show that  $u_1(x)$  and  $u_2(x)$  can be chosen such that (2) is a solution of Equation (1). Since we have to determine two functions  $u_1(x)$  and  $u_2(x)$ , we will have to specify two conditions. Obviously, one of these is that  $y_p$  satisfies Equation (1). The other is to be chosen so as to facilitate the calculations. Differentiating Equation (2) with respect to  $x$ , we get

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x) + u_1'(x)y_1(x) + u_2'(x)y_2(x).$$

In order to simplify the calculations, we shall impose the condition

$$u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0. \quad (3)$$

This implies

$$y_p'(x) = u_1(x)y_1'(x) + u_2(x)y_2'(x).$$

So  $y_p''(x) = u_1(x)y_1''(x) + u_2(x)y_2''(x) + u_1'(x)y_1'(x) + u_2'(x)y_2'(x).$

Substituting  $y_p(x)$ ,  $y_p'(x)$ , and  $y_p''(x)$  in Equation (1), we get

$$a_0(u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2') + a_1(u_1 y_1' + u_2 y_2') + a_2(u_1 y_1 + u_2 y_2) = g(x). \quad (4)$$

Since  $y_1$  and  $y_2$  are solutions of the associated homogeneous equation of (1), we get

$$a_0 y_1'' + a_1 y_1' + a_2 y_1 = 0 \text{ and } a_0 y_2'' + a_1 y_2' + a_2 y_2 = 0.$$

So Equation (4) becomes

$$a_0(u_1' y_1' + u_2' y_2') = g(x)$$

or

$$u_1' y_1' + u_2' y_2' = g(x)/a_0. \quad (5)$$

Equations (3) and (5) give two conditions governing the choice of  $u_1$  and  $u_2$ . If we can choose  $u_1$  and  $u_2$  so as to satisfy these two conditions, then

we are through. Writing these together, we have

$$\begin{aligned}u_1'(x)y_1(x) + u_2'(x)y_2(x) &= 0 \\u_1'(x)y_1'(x) + u_2'(x)y_2'(x) &= g(x)/a_0.\end{aligned}$$

Solving for  $u_1(x)$  and  $u_2(x)$  by Cramer's rule, we have

$$u_1(x) = \frac{\begin{vmatrix} 0 & y_2(x) \\ g(x)/a_0 & y_2'(x) \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = -\frac{y_2(x)g(x)}{a_0W[y_1(x), y_2(x)]} \quad (6)$$

and

$$u_2(x) = \frac{\begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & g(x)/a_0 \end{vmatrix}}{\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}} = \frac{y_1(x)g(x)}{a_0W[y_1(x), y_2(x)]}. \quad (7)$$

Carrying out the integrations, we get

$$\begin{aligned}u_1(x) &= -\int^x \frac{g'(t)y_2(t)dt}{a_0W[y_1(t), y_2(t)]} \\u_2(x) &= \int^x \frac{g(t)y_1(t)dt}{a_0W[y_1(t), y_2(t)]}.\end{aligned}$$

Thus, a particular solution of Equation (1) is

$$\begin{aligned}y_p(x) &= -y_1(x) \int^x \frac{g(t)y_2(t)dt}{a_0W[y_1(t), y_2(t)]} + y_2(x) \int^x \frac{g(t)y_1(t)dt}{a_0W[y_1(t), y_2(t)]} \\&= \int^x \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{a_0W[y_1(t), y_2(t)]} g(t)dt.\end{aligned} \quad (8)$$

Hence, the complete solution of Equation (1) is

$$y_1 = C_1y_1(x) + C_2y_2(x) + \int^x \frac{y_2(x)y_1(t) - y_1(x)y_2(t)}{a_0W[y_1(t), y_2(t)]} g(t)dt.$$

**A2.1 Remark** The same method is applicable to equations of arbitrary order. The only modification is that in the case of the  $n$ -th order equation we shall need  $n$  arbitrary functions  $u_1(x), u_2(x), \dots, u_n(x)$  in  $y_p$ , since there are  $n$  arbitrary constants  $C_1, C_2, \dots, C_n$  in the complementary function. We shall impose on it  $(n-1)$  conditions, one at each successive differentiation, in addition to the first condition that  $y_p = u_1y_1 + u_2y_2 + \dots + u_ny_n$  satisfies the nonhomogeneous differential equation.

Instead of carrying out these details in the general case, we shall illustrate the method by working out some suitable examples.

**Example A.6**  $(D^3 - 2D - 1)y = \sin x$ .

The complementary function is (cf Example A.1)

$$y_0 = C_1e^{(1+\sqrt{2})x} + C_2e^{(1-\sqrt{2})x}.$$

So we assume

$$y_p = u_1(x)e^{(1+\sqrt{2})x} + u_2(x)e^{(1-\sqrt{2})x}.$$

We can now obtain  $u_1(x)$  and  $u_2(x)$  by going through the process already described. Since we have done this for the general equation of the second order, we shall merely plug in the functions in Equation (8). We have

$$W[y_1(x), y_2(x)] = W[e^{(1+\sqrt{2})x}, e^{(1-\sqrt{2})x}] = -2\sqrt{2}e^{2x}.$$

Hence

$$\begin{aligned} y_p(x) &= \int^x \frac{e^{(1-\sqrt{2})x} e^{(1+\sqrt{2})t} - e^{(1-\sqrt{2})t} e^{(1+\sqrt{2})x}}{-2\sqrt{2}e^{2t}} \sin t \, dt \\ &= 2\sqrt{2} \int^x [e^{(1+\sqrt{2})x} e^{-(\sqrt{2}+1)t} - e^{(1-\sqrt{2})x} e^{(\sqrt{2}-1)t}] \sin t \, dt \\ &= -\frac{1}{4} (\sin x - \cos x). \end{aligned}$$

**A2.2 Remark** The reader is advised not to use result (8) for finding  $y_p$ . He should carry out all the steps needed to arrive at Equation (8) as illustrated again in Example A.7.

*Example A 7*  $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 2x^2 + 4 \sin x.$

The complementary function is (cf Example A 4)

$$y_c = C_1 + C_2 \cos x + C_3 \sin x.$$

So we assume

$$y_p(x) = u_1(x) + u_2(x) \cos x + u_3(x) \sin x.$$

Then

$$y_p'(x) = u_1'(x) + u_2'(x) \cos x + u_3'(x) \sin x - u_2(x) \sin x + u_3(x) \cos x.$$

When we impose the condition

$$u_1'(x) + u_2'(x) \cos x + u_3'(x) \sin x = 0, \quad (9)$$

we get

$$y_p'(x) = -u_2(x) \sin x + u_3(x) \cos x.$$

Differentiating again, we get

$$y_p''(x) = -u_2'(x) \cos x - u_2(x) \sin x - u_3'(x) \sin x + u_3(x) \cos x.$$

We now impose another condition

$$-u_2'(x) \sin x + u_3'(x) \cos x = 0. \quad (10)$$

So

$$y_p''(x) = -u_2(x) \cos x - u_3(x) \sin x$$

and  $y_p'''(x) = -u_2'(x) \cos x - u_2(x) \sin x + u_3'(x) \sin x - u_3(x) \cos x.$

The third condition on  $u_1(x)$ ,  $u_2(x)$ ,  $u_3(x)$  is that  $y_p$  should satisfy the equation

$$y_p''' + y_p' = 2x^2 + 4 \sin x.$$

Substituting  $y_p$ , we obtain

$$-u_2'(x) \cos x - u_3'(x) \sin x = 2x^2 + 4 \sin x. \quad (11)$$

Writing Equations (9), (10), and (11) together, we get

$$\begin{aligned}u_1'(x) + u_2(x) \cos x + u_3'(x) \sin x &= 0 \\-u_2'(x) \sin x + u_3'(x) \cos x &= 0 \\-u_2'(x) \cos x - u_3'(x) \sin x &= 2x^2 + 4 \sin x.\end{aligned}$$

Solving by Cramer's rule, we obtain

$$\begin{aligned}u_1'(x) &= 2x^2 + 4 \sin x, \quad u_2(x) = -(2x^2 + 4 \sin x) \cos x \\ \text{and} \quad u_3(x) &= -(2x^2 + 4 \sin x) \sin x.\end{aligned}$$

This, on integration, gives

$$u_1(x) = \frac{2x^3}{3} - 4 \cos x \quad (12)$$

$$u_2(x) = -2 \sin^2 x - 2x^2 \sin x - 4x \cos x + 4 \sin x \quad (13)$$

and

$$u_3(x) = 2x^2 \cos x - 4x \sin x - 4 \cos x - 2x + \sin 2x. \quad (14)$$

$$\begin{aligned}\text{Thus} \quad y_p(x) &= \frac{2x^3}{3} - 4 \cos x + u_2(x) \cos x + u_3(x) \sin x \\ &= \frac{2x^3}{3} - 4 \cos x - 4x - 2x \sin x.\end{aligned}$$

Hence, the complete solution is

$$\begin{aligned}y &= C_1 + C_2 \cos x + C_3 \sin x + \frac{2x^3}{3} - 4x - 4 \cos x - 2x \sin x \\ &= C_1 + D_2 \cos x + C_3 \sin x + \frac{2}{3} x^3 - 4x - 2x \sin x.\end{aligned}$$

Note that in this example we needed three conditions, because the differential equation involved is of order 3.

### Problem Set A2

- Using the method of variation of parameters, find the general solution for the following differential equations :

$$(a) (D^2 + 4)y = x \sin 2x \quad (b) (D^2 - 4D + 4)y = xe^{-2x}$$

$$(c) (D^2 + 4)y = \sec 2x, \quad (d) (3D^2 + D - 2)y = 3e^x \\ 0 < x < \pi/4$$

$$(e) (4D^2 + 1)y = \sec^2 x/2, \quad (f) (D^2 + 1)y = \sin x \\ 0 < x < \pi$$

$$(g) (D^2 + 4D + 4)y = xe^{-2x} \quad (h) (6D^2 + D - 2)y = xe^{-x}$$

$$(i) (D^3 + D)y = x^2 \quad (j) (D^3 - D)y = x \cos x$$

$$(k) y''' - 3y'' - y' + 3y = e^{3x} \quad (l) 2y''' + y'' - y' = e^{x/2}.$$

- Let  $y_1(x), y_2(x), \dots, y_n(x)$  be  $n$  linearly independent solutions of the associated homogeneous equation of

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n)y = g(x).$$

Use the method of variation of parameters to prove that

$$y_p(x) = \int^x K(x, t) \frac{g(t)}{a_0} dt, \text{ where}$$

$$K(x, t) = \frac{1}{W[y_1(t), y_2(t), \dots, y_n(t)]} \begin{vmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)}(t) & y_2^{(n-2)}(t) & \dots & y_n^{(n-2)}(t) \\ y_1(x) & y_2(x) & \dots & y_n(x) \end{vmatrix}.$$

### A3 METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is applicable whenever the right-hand member  $g(x)$  of the nonhomogeneous  $n$ -th order linear differential equation with constant coefficients

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n)y = g(x) \quad (1)$$

is a finite linear combination of

- (i) powers of  $x$ ;
- (ii)  $\sin \alpha x$ ,  $\cos \alpha x$ ;
- (iii)  $e^{\alpha x}$ ; and
- (iv) finite products of any of the functions in (i), (ii), and (iii).

In order to outline this method, we need the following definition.

**A3.1 Definition** Given a function  $f(x)$ , a set of those linearly independent functions whose finite linear combinations give the function  $f$  and all its derivatives is called a *derivative family* of  $f(x)$ . It is denoted by  $D[f]$ .

**Example A.8** Here we list the derivative families of four functions :

- (a)  $D[x^m] = \{x^m, x^{m-1}, \dots, x^2, x, 1\}$
- (b)  $D[e^{\beta x}] = \{e^{\beta x}\}$
- (c)  $D[\sin \alpha x] = \{\sin \alpha x, \cos \alpha x\}$
- (d)  $D[\cos \beta x] = \{\cos \beta x, \sin \beta x\}$ .

We shall be interested in only those functions  $f$  for which  $D[f]$  is a finite set; for example, the functions listed in Example A.8 and their finite products, namely,  $x^m$ ,  $e^{\alpha x}$ ,  $\sin \alpha x$ ,  $\cos \alpha x$ ,  $x^m \sin \alpha x$ ,  $x^m \cos \alpha x$ ,  $x^m e^{\alpha x}$ ,  $e^{\alpha x} \sin \beta x$ ,  $e^{\alpha x} \cos \beta x$ ,  $x^m e^{\alpha x} \sin \beta x$ , and  $x^m e^{\alpha x} \cos \beta x$ .

For each of these functions, a derivative family can be obtained as illustrated in Example A.9.

**Example A.9**  $D[e^{\alpha x} \cos \beta x] = \{f(x)g(x) \mid f(x) \in D[e^{\alpha x}], g(x) \in D[\cos \beta x]\}$   
 $= \{e^{\alpha x} \cos \beta x, e^{\alpha x} \sin \beta x\}.$



$$D[x^m \sin ax] = \{f(x)g(x) \mid f(x) \in D[x^m], g(x) \in D[\sin ax]\} \\ = \{x^m \sin ax, x^{m-1} \sin ax, \dots, x \sin ax, \sin ax, \\ x^m \cos ax, x^{m-1} \cos ax, \dots, x \cos ax, \cos ax\}.$$

$$D[x^m e^{ax} \sin bx] = \{f(x)g(x)h(x) \mid f(x) \in D[x^m], \\ g(x) \in D[e^{ax}], h(x) \in D[\sin bx]\}.$$

**A3.2 Remark** It can be easily verified that for two functions  $p(x)$  and  $q(x)$

$$D[p(x) + q(x)] = D[p(x)] \cup D[q(x)].$$

Now we are ready to enunciate the method of undetermined coefficients for finding the particular integral  $y_p$  of

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_n)y = g(x).$$

**Step 1** Write

$$g(x) = g_1(x) + g_2(x) + \dots + g_k(x),$$

where each  $g_i(x)$  is a function of the type mentioned immediately after Example A.8.

**Step 2** Find the derivative family  $D[g_i]$  for each  $i = 1, 2, \dots, k$ .

**Step 3** Find the complementary function  $y_c$  of the differential equation in question. Let

$$y_c = C_1 y_1(x) + C_2 y_2(x) + \dots + C_n y_n(x).$$

For each  $i = 1, \dots, k$ , check whether any member of  $D[g_i]$  is already one of the  $y_j$ 's, i.e. a solution of the associated homogeneous equation. If so, then multiply each member of  $D[g_i]$ , for that  $i$ , by the least power of  $x$  such that the new set thus obtained does not contain any  $y_j$ . Call this new set the modified  $D[g_i]$ .

If no member of  $D[g_i]$  is a  $y_j$ , then leave that  $D[g_i]$  unchanged.

**Step 4** Let  $S[g]$  be the union of all those  $D[g_i]$ 's that have not been modified and all the modified  $D[g_i]$ 's.

**Step 5** Assume

$$y_p = \sum A_i h_i(x),$$

where the  $A_i$ 's are arbitrary constants, and  $h_i(x)$  varies through all the members of  $S[g]$ . In other words,  $y_p$  is a linear combination of all members of  $S[g]$ .

**Step 6** Substitute  $y_p$  in the differential equation and determine the coefficients  $A_i$ 's by equating coefficients of identical terms on both sides.

We illustrate this method through the following examples.

**Example A.10**  $(D^3 - 2D - 1)y = \sin x$ .

The complementary function is (cf Example A.1)

$$y_c = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x},$$

To get  $y_p$ , we proceed as follows :

**Step 1** It is not necessary here.

**Step 2**  $D[\sin x] = \{\sin x, \cos x\}$ .

**Step 3** No modification of  $D[\sin x]$  is needed, because neither  $\sin x$  nor  $\cos x$  occurs in the complementary function.

**Step 4**  $S[\sin x] = \{\sin x, \cos x\}$ .

**Step 5** Assume  $y_p(x) = A \sin x + B \cos x$ .

**Step 6**  $y'_p(x) = A \cos x - B \sin x$ ,  
 $y''_p(x) = -A \sin x - B \cos x$ .

Substitution in the equation gives

$$(2B - 2A) \sin x - (2A + 2B) \cos x = \sin x.$$

Therefore,  $2A + 2B = 0$  and  $2B - 2A = 1$ . This gives  $A = -B = -1/4$ . Hence,  $y_p(x) = (\cos x - \sin x)/4$ . Thus, the complete solution is

$$y = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x} - (\sin x - \cos x)/4.$$

**Example A.11**  $(4D^2 + 12D + 9)y = xe^{-3x/2}$ . The complementary function is (cf Example A.2)

$$y_c = C_1 e^{-3x/2} + C_2 x e^{-3x/2}.$$

**Step 1** It is not necessary here.

**Step 2**  $D[xe^{-3x/2}] = \{xe^{-3x/2}, e^{-3x/2}\}$ .

**Step 3** Both the functions in the set  $D[xe^{-3x/2}]$  appear in the complementary function. So we multiply them by  $x^2$ . (Note that multiplication by  $x$  will leave one function in the set to coincide with one in the complementary function.) So the modified derivative family is  $\{x^3 e^{-3x/2}, x^2 e^{-3x/2}\}$ .

**Step 4**  $S[xe^{-3x/2}] = \{x^3 e^{-3x/2}, x^2 e^{-3x/2}\}$ .

**Step 5** Assume  $y_p = Ax^3 e^{-3x/2} + Bx^2 e^{-3x/2}$ .

**Step 6**  $y'_p = e^{-3x/2} \left( -\frac{3A}{2} x^3 + 3Ax^2 - \frac{3B}{2} x^2 + 2Bx \right)$   
 $y''_p = e^{-3x/2} \left( \frac{9A}{4} x^3 - \frac{9A}{2} x^2 + \frac{9B}{4} x^2 - 3Bx - \frac{9A}{2} x^2 + 6Ax - 3Bx + 2B \right).$

Substitution in the equation gives

$$e^{-3x/2} (24Ax + 8B) = x e^{-3x/2}.$$

Hence,  $B = 0$  and  $A = 1/24$ . Thus,  $y_p(x) = (x^3/24)e^{-3x/2}$  and the complete solution is

$$y = (C_1 + C_2 x + x^3/24)e^{-3x/2}.$$

**Example A.12**  $(D^2 - 4D + 13)y = x^2 + \cos 2x$ . The complementary function (real) is (cf Example A.3)

$$y_0 = e^{2x}(C_1 \cos 3x + C_2 \sin 3x).$$

*Step 1*  $g(x) = x^2 + \cos 2x = g_1(x) + g_2(x)$ ,

where  $g_1(x) = x^2$ ,  $g_2(x) = \cos 2x$ .

*Step 2*  $D[g_1] = \{x^2, x, 1\}$ ,  $D[g_2] = \{\cos 2x, \sin 2x\}$ .

*Step 3* No modification of the derivative families is needed.

*Step 4*  $S[g] = \{x^3, x, 1, \sin 2x, \cos 2x\}$ .

*Step 5* Assume  $y_p(x) = Ax^3 + Bx + C + D \sin 2x + E \cos 2x$ .

*Step 6* Substitution of  $y_p$  in the equation gives  $(2A - 4B + 13C) + (-8A + 13B)x + 13Ax^3 + (13E - 8D - 4E) \cos 2x + (13D + 8E - 4D) \sin 2x = x^2 + \cos 2x$ .

This gives  $A = 1/13$ ,  $B = 8/169$ ,  $C = 6/2197$ ,  $D = -8/145$ ,  $E = 9/145$ .

Hence,

$$y_p = \frac{x^3}{13} + \frac{8x}{169} + \frac{6}{2197} + \frac{9 \cos 2x - 8 \sin 2x}{145},$$

and the complete solution is

$$y = e^{2x}(C_1 \cos 3x + C_2 \sin 3x) + y_p.$$

*Example A.13*  $(D^3 + D)y = x^2 + 2e^x \sin x$ .

The complementary function is

$$y_0 = C_0 + C_1 \cos x + C_2 \sin x \quad (\text{cf. Example A.4}).$$

*Step 1*  $g(x) = g_1(x) + g_2(x)$ , where  $g_1(x) = x^2$ ,  $g_2(x) = 2e^x \sin x$ .

*Step 2*  $D[g_1] = \{x^2, x, 1\}$ ,  $D[g_2] = \{e^x \sin x, e^x \cos x\}$ .

*Step 3* Since '1' is included in the complementary function,  $D[x^2]$  is modified as  $\{x^3, x^2, x\}$ .

*Step 4*  $S[g] = \{x^3, x^2, x, e^x \sin x, e^x \cos x\}$ .

*Step 5* Assume  $y_p = Ax^3 + Bx^2 + Cx + De^x \sin x + Ee^x \cos x$ .

*Step 6* Substituting  $y_p$  in the equation, we get

$$6A + 3Ax^3 + 2Bx + C - (D + 3E)e^x \sin x + (3D - E)e^x \cos x = x^2 + 2e^x \sin x.$$

Thus,  $A = 1/3$ ,  $B = 0$ ,  $C = -2$ ,  $D = -1/5$ ,  $E = -3/5$ .

Hence,

$$y_p = \frac{x^3}{3} - 2x - \frac{1}{5}e^x \sin x - \frac{3}{5}e^x \cos x,$$

and the general solution is

$$y = C_0 + C_1 \cos x + C_2 \sin x + \frac{x^3}{3} - 2x - \frac{e^x \sin x}{5} - \frac{3e^x \cos x}{5}.$$

### Problem Set A3

1. Using the method of undetermined coefficients, find the general solution for the following differential equations :

- (a)  $y'' - 2y' - 3y = e^{2x}$  (b)  $y'' + 4y = \sin 2x$   
 (c)  $y'' + 2y' + 2y = xe^x + x^2$  (d)  $(D^2 + 2D - 1)y = 2 \cosh x$   
 (e)  $(D^3 - D^2 - D + 1)y = \sin x + 1 + xe^{-x}$   
 (f)  $(D^3 + 9D)y = x$   
 (g)  $(D^4 + D^3 - D - 1)y = x \sinh x + x^2 + 3$   
 (h)  $(D^3 + D)y = 2 \cos x$   
 (i)  $(D^4 + 4D^3 - 5D^2)y = x + x \sin x$   
 (j)  $y''' - y'' - 6y' = xe^{-2x} + \sin x + x + e^{-3x}$   
 (k)  $(D^2 + 4D + 11)y = xe^{-2x} + e^{-2x} \sin 3x.$

## A4 OPERATIONAL METHODS

The two methods discussed in § A2 and A3, namely, the method of variation of parameters and the method of undetermined coefficients, suffice to take care of all problems of finding a particular solution. However, in this article we add to the reader's armour another tool, which often comes in very useful: Consider the equation

$$f(D)y = g(x), \quad (1)$$

where  $f(D) = a_0D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n$ , and  $a_0, a_1, \dots, a_n$  are constants. To obtain one particular solution of this equation, we are tempted to write formally

$$y = \frac{1}{f(D)} g(x). \quad (2)$$

This is only a formal step, which would be useful only if it enables us to calculate  $y$ . First, we have to give a meaning to Equation (2). Second, we have to relate Equation (2) to Equation (1). We achieve both purposes simultaneously by defining Equation (2) as follows.

**A4.1 Definition** If  $f(D) = a_0D^n + a_1D^{n-1} + \dots + a_n$ , where  $a_0, a_1, \dots, a_n$  are constants, then

$$\frac{1}{f(D)} g(x)$$

is defined as a function  $y$  for which  $f(D)y = g(x)$ .

This definition gives meaning to Equation (2) and also relates it to Equation (1).

We shall now show that  $\frac{1}{f(D)} g(x)$  can indeed be calculated easily in certain favourable situations.

The simplest case occurs when  $f(D) = D$ . Now  $\frac{1}{D} g(x)$  is that function  $y$  for which  $Dy = g(x)$ . This means

$$\frac{dy}{dx} = g(x).$$

So  $y = \int g(x)dx$ . Since we are interested in only one particular solution, we shall ignore the arbitrary constant. Thus, we have

$$\frac{1}{D} g(x) = \int g(x)dx. \quad (3)$$

Extending this we see that

$$\frac{1}{D^2} g(x) = \int (\int g(x)dx)dx, \quad (4)$$

and so on

We now take the case  $f(D) = D - \alpha$ . Then  $\frac{1}{D - \alpha} g(x)$  is evaluated as follows: Let

$$y = \frac{1}{D - \alpha} g(x).$$

Then  $(D - \alpha)y = g(x)$ , which is a linear differential equation. So

$$y = e^{\alpha x} \int e^{-\alpha x} g(x)dx \quad (\text{cf } \S 4.9).$$

Thus, we have

$$\frac{1}{D - \alpha} g(x) = e^{\alpha x} \int e^{-\alpha x} g(x)dx. \quad (5)$$

*Example A.14* Calculate  $\frac{1}{D - \alpha} e^{\alpha x} \sin \beta x$ . Using Equation (5), we get

$$\begin{aligned} \frac{1}{D - \alpha} e^{\alpha x} \sin \beta x &= e^{\alpha x} \int e^{-\alpha x} e^{\alpha x} \sin \beta x dx \\ &= -\frac{e^{\alpha x} \cos \beta x}{\beta}. \end{aligned}$$

To handle more complicated expressions for  $f(D)$ , we first note the following result.

**A4.2 Theorem** *The set of all operator polynomials  $f(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n I$ , where  $a_0, a_1, \dots, a_n$  are in general complex constants, is a vector space (real or complex, according to the scalars used). Here  $I$  is the identity operator.*

The proof is left to the reader. Note that each such  $f(D)$  is itself a linear operator on  $\mathcal{C}^{(\infty)}$ .

If  $f(D)$  and  $g(D)$  are operator polynomials, then  $f(D) \circ g(D)$  or simply  $f(D)g(D)$  is defined, just as we defined  $S \circ T$  in Chapter 1, i.e.,  $(f(D)g(D))y = f(D)(g(D)y)$ . This multiplication is

(a) *associative*, i.e.

$$f(D)((g(D)h(D)) = (f(D)g(D))h(D),$$

(b) *distributive over addition*, i.e.

$$f(D)(g(D) + h(D)) = f(D)g(D) + f(D)h(D)$$

and  $(f(D) + g(D))h(D) = f(D)h(D) + g(D)h(D),$

(c) *commutative*, i.e.

$$f(D)g(D) = g(D)f(D).$$

**A4.3 Remark** It may be noted that the multiplication of operator polynomials is not commutative if the coefficients are functions of  $x$ .

For example,

$$(xD + I)D = xD^2 + D \text{ but } D(xD + I) = xD^2 + 2D.$$

Because of the aforesaid properties, we may use the results of ordinary algebra pertaining to factorisation of polynomials. Note that  $D - I$  can be written as  $D - 1$  without any damage to the working. In general,

$$(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n I$$

can be written as

$$f(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n,$$

as we have been doing all along. In particular, we can factorise  $D^2 - 5D + 6$  as  $(D - 2)(D - 3)$ .

With this background, let us now handle  $\frac{1}{f(D)} g(x)$ . We illustrate the method through the following example.

**Example A.15** Evaluate  $\frac{1}{D^2 - 1} \sin x$ .

We write

$$\begin{aligned} \frac{1}{D^2 - 1} \sin x &= \frac{1}{(D - 1)} \left( \frac{1}{D + 1} \sin x \right) \\ &= \frac{1}{D - 1} (e^{-x} \int e^x \sin x \, dx) && \text{(by (5))} \\ &= \frac{1}{2} \frac{1}{D - 1} (\sin x - \cos x) \\ &= \frac{1}{2} e^x \int e^{-x} (\sin x - \cos x) dx && \text{(by (5))} \\ &= -\frac{1}{2} \sin x. \end{aligned}$$

To solve such problems we can also use partial fractions as follows :

$$\begin{aligned} \frac{1}{D^2 - 1} \sin x &= \frac{1}{(D - 1)(D + 1)} \sin x \\ &= \frac{1}{2} \left( \frac{1}{D - 1} - \frac{1}{D + 1} \right) \sin x \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{1}{D-1} \sin x - \frac{1}{2} \frac{1}{D+1} \sin x \\
&= \frac{1}{2} e^x \int e^{-x} \sin x \, dx - \frac{1}{2} e^{-x} \int e^x \sin x \, dx.
\end{aligned}$$

This, on simplification, gives

$$\frac{1}{D^2-1} \sin x = -\frac{1}{2} \sin x.$$

Actually, there should be two arbitrary constants, but we omit them since we are interested in only a particular integral.

In such operator methods the following theorem is useful.

**A4.4 Theorem** If  $f(D)$  is the operator polynomial  $a_0 D^n + a_1 D^{n-1} + \dots + a_n$  with constant coefficients, then

- (a)  $f(D)e^{\alpha x} = f(\alpha)e^{\alpha x}$ .
- (b)  $f(D)(e^{\alpha x}g(x)) = e^{\alpha x}f(D + \alpha)g(x)$ .
- (c)  $f(D^2) \sin \alpha x = f(-\alpha^2) \sin \alpha x$ .
- (d)  $f(D^2) \cos \alpha x = f(-\alpha^2) \cos \alpha x$ .

The proof of this theorem is left to the reader. Using Theorem A4.4 and Definition A4.1, the next theorem immediately follows.

**A4.5 Theorem** Let  $f(D) = a_0 D^n + a_1 D^{n-1} + \dots + a_n$  be an operator polynomial with constant coefficients. Then

- (a)  $\frac{1}{f(D)} e^{\alpha x} = \frac{e^{\alpha x}}{f(\alpha)}, f(\alpha) \neq 0$ .
- (b)  $\frac{1}{f(D)} e^{\alpha x} g(x) = e^{\alpha x} \frac{1}{f(D + \alpha)} g(x)$ .
- (c)  $\frac{1}{f(D^2)} \sin \alpha x = \frac{1}{f(-\alpha^2)} \sin \alpha x, f(-\alpha^2) \neq 0$ .
- (d)  $\frac{1}{f(D^2)} \cos \alpha x = \frac{1}{f(-\alpha^2)} \cos \alpha x, f(-\alpha^2) \neq 0$ .
- (e)  $\frac{1}{f(D)} (P(x) + Q(x)) = \frac{1}{f(D)} P(x) + \frac{1}{f(D)} Q(x)$ .
- (f)  $\frac{1}{f(D)g(D)} P(x) = \frac{1}{f(D)} \left( \frac{1}{g(D)} P(x) \right)$ .

With this theorem, let us find particular integrals  $y_p$  for the differential equations in the following example, some of which we have already worked out by earlier methods.

**Example A.16** Consider the differential equations

- (a)  $(D^2 - 2D - 1)y = \sin x$
- (b)  $(4D^2 + 12D + 9)y = xe^{-3x/2}$
- (c)  $(D^2 - 4D + 13)y = x^2 + \cos 2x$

$$(d) \quad (D^3 + D)y = 2x^2 + 4 \sin x.$$

$$\begin{aligned}
 (a) \quad y_p &= \frac{1}{D^3 - 2D - 1} \sin x \\
 &= \frac{D^3 + 2D - 1}{(D^3 - 1)^2 - 4D^2} \sin x \\
 &= (D^3 + 2D - 1) \left( \frac{1}{(D^3 - 1)^2 - 4D^2} \sin x \right) \\
 &= \frac{1}{8} (D^3 + 2D - 1) \sin x \quad (\text{by Theorem A4.5}) \\
 &= \frac{1}{4} (\cos x - \sin x).
 \end{aligned}$$

It may be noted that, as a working rule, we can use also the following method :

$$\begin{aligned}
 y_p &= \frac{1}{D^3 - 2D - 1} \sin x \\
 &= \frac{1}{-1 - 2D - 1} \sin x \quad (\text{by writing } -1 \text{ for } D^3) \\
 &= -\frac{1}{2} \frac{1}{D + 1} \sin x \\
 &= -\frac{1}{2} \frac{D - 1}{D^2 - 1} \sin x = -\frac{1}{2} (D - 1) \left( \frac{1}{D^2 - 1} \sin x \right) \\
 &= \frac{1}{4} (D - 1) \sin x \quad (\text{by Theorem A4.5}) \\
 &= -\frac{1}{4} (\sin x - \cos x).
 \end{aligned}$$

Note that in this method we replaced  $D^3$  by  $-1$ . In general, to evaluate  $\frac{1}{f(D)} \sin ax$  or  $\frac{1}{f(D)} \cos ax$ , we replace  $D^2$  by  $-a^2$  in  $f(D)$  provided  $(D^2 + a^2)$  is not a factor of  $f(D)$ .

$$\begin{aligned}
 (b) \quad y_p &= \frac{1}{4D^3 + 12D + 9} x e^{-3x/2} = \frac{1}{(2D + 3)^2} x e^{-3x/2} \\
 &= e^{-3x/2} \frac{1}{4D^2} x \quad (\text{by Theorem A4.5}) \\
 &= \frac{x^2 e^{-3x/2}}{24}.
 \end{aligned}$$

$$\begin{aligned}
 (c) \quad y_p &= \frac{1}{D^3 - 4D + 13} (x^2 + \cos 2x) \\
 &= \frac{1}{D^3 - 4D + 13} x^2 + \frac{1}{D^3 - 4D + 13} \cos 2x.
 \end{aligned}$$

Taking the second part, we have

$$\frac{1}{D^3 - 4D + 13} \cos 2x = \frac{1}{9 - 4D} \cos 2x$$



$$\begin{aligned}
&= \frac{9 + 4D}{81 - 16D^2} \cos 2x \\
&= (9 + 4D) \left( \frac{1}{81 - 16D^2} \cos 2x \right) \\
&= \frac{(9 + 4D)}{145} \cos 2x = \frac{9 \cos 2x - 8 \sin 2x}{145}.
\end{aligned}$$

To tackle the first part, we have no operational method so far. But the following 'formal' method, for which we cannot give any justification within the scope of this appendix, is successful :

$$\begin{aligned}
\frac{1}{D^2 - 4D + 13} x^2 &= \frac{1}{13(1 + \frac{D^2 - 4D}{13})} x^2 \\
&= \frac{1}{13} \left( 1 - \frac{D^2 - 4D}{13} + \frac{(D^2 - 4D)^2}{169} + \dots \right) x^2 \\
&\quad \text{(by a formal long division)} \\
&= \frac{1}{13} \left( x^2 - \frac{2 - 8x}{13} + \frac{32}{169} \right) \\
&= \frac{1}{13} \left( x^2 + \frac{8x}{13} + \frac{6}{169} \right).
\end{aligned}$$

Hence,

$$y_p = \frac{x^2}{13} + \frac{8x}{169} + \frac{6}{2197} + \frac{9 \cos 2x - 8 \sin 2x}{145}.$$

**A4.6 Remark** The working of  $\frac{1}{D^2 - 4D + 13} x^2$ , though not justifiable at this stage, can be very powerful. The reader is urged to use it with the full knowledge that he is using only a formal method, the validity of which is beyond the scope of this appendix.

In any case the reader will have already noted that wherever the operational method can be used it is really powerful.

$$\begin{aligned}
\text{(d)} \quad y_p &= \frac{1}{D^2 + D} (4 \sin x + 2x^2) \\
&= \frac{1}{D^2 + D} (4 \sin x) + \frac{1}{D^2 + D} (2x^2) \\
&= y_{p_1} + y_{p_2}.
\end{aligned}$$

$$\begin{aligned}
\text{Here} \quad y_{p_1} &= 4 \frac{1}{D(D^2 + 1)} \sin x = 4 \frac{1}{(D^2 + 1)} \left( \frac{1}{D} \sin x \right) \\
&= 4 \frac{1}{D^2 + 1} (-\cos x) \\
&= -4 \frac{1}{D^2 + 1} \cos x.
\end{aligned}$$

We cannot apply Theorem A4.5. In this case we continue as follows :

$$\begin{aligned}
 y_{p_1} &= -4 \operatorname{Re} \left( \frac{1}{D^2 + 1} e^{ix} \right) \\
 &= -4 \operatorname{Re} \left( e^{ix} \frac{1}{(D + i)^2 + 1} \right) \\
 &= -4 \operatorname{Re} \left( e^{ix} \frac{1}{(D + 2i)D} \right) \\
 &= -4 \operatorname{Re} \left( e^{ix} \frac{1}{D + 2i} x \right) \\
 &= -4 \operatorname{Re} \left( \frac{e^{ix}}{2i} \left( 1 - \frac{D}{2i} + \frac{D^2}{4i^2} - \dots \right) x \right) \\
 &= -4 \operatorname{Re} \left( e^{ix} \left( -\frac{ix}{2} + \frac{1}{4} \right) \right) \\
 &= -4 \left( \frac{1}{4} \cos x + \frac{x \sin x}{2} \right) = -2x \sin x - \cos x .
 \end{aligned}$$

Now

$$\begin{aligned}
 y_{p_2} &= \frac{1}{D^3 + D} (2x^2) = \frac{1}{D^2 + 1} \left( \frac{1}{D} (2x^2) \right) \\
 &= \frac{2}{3} \frac{1}{D^2 + 1} x^3 = \frac{2}{3} (1 - D^2 + D^4 - \dots) x^3 \\
 &= \frac{2}{3} (x^3 - 6x) = \frac{2x^3}{3} - 4x .
 \end{aligned}$$

Thus,

$$y_p = -\cos x - 2x \sin x + \frac{2x^3}{3} - 4x .$$

It may be noted that the term  $(-\cos x)$  can be omitted in view of the fact that  $\cos x$  is a part of  $y_c$ .

### Problem Set A4

- Factorise the operator in each of the following cases and hence find a particular integral :
  - $(D^2 - 1)y = e^x$
  - $(D^2 - 3D + 2)y = x + e^{2x}$
  - $(D^3 + 4D^2 - 5D)y = x + \sin x$
  - $(8D^3 + 12D^2 - 2D - 3)y = x + xe^x$
- Using operational methods, determine a particular integral for the following differential equations :
  - $(D^3 - 3D^2 + 7D + 5)y = e^{-2x}$
  - $(D^3 + 9D)y = \cos 3x$
  - $(D^3 + 3D + 1)y = e^x \sin 2x$

(d)  $(D^2 + 4D)y = \sin 2x$

(e)  $(D^2 + 3D - 4)y = e^x(3x^2 + 5x - 7)$

(f)  $(D - 2)^2 y = e^{2x} \cos x$

(g)  $(2D^3 + 5D^2 - 3D)y = e^{-3x}$

(h)  $(D^2 - 4D + 4)^2 y = x^3 e^{2x}$ .

3. Prove Theorem A4.2.

4. Prove Theorem A4.4.

5. Prove Theorem A4.5.

# Answers to Problems

## Problem Set 1.1

4. (a)  $\{x \in R \mid -1 < x < 1\}$  (b)  $\{x \in R \mid -1 < x < 0\}$   
 (c)  $\{x \in R \mid -1 < x < 2\}$  (d)  $\{x \in R \mid x \geq 2 \text{ or } x < -2\}$   
 (e)  $\{x \in R \mid x \geq 1 \text{ or } x < -1\}$  (f)  $\phi$ .
5. (a) The set of all complex numbers represented by the points inside a circle of radius 4 centred at the origin  
 (b) The set of all complex numbers represented by the points on a circle of radius  $4/3$  centred at  $(-5/3, 0)$   
 (c)  $\{z \in C \mid \text{Im } z \leq 0\}$   
 (d) The set of all complex numbers represented by the points on a circle of radius 3 centred at  $(-2, 0)$ .
6. (a)  $\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}$   
 (b)  $\phi, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\alpha, \delta\}, \{\beta, \gamma\}, \{\beta, \delta\}, \{\gamma, \delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \beta, \delta\}, \{\alpha, \gamma, \delta\}, \{\beta, \gamma, \delta\}, \{\alpha, \beta, \gamma, \delta\}$ .
7.  $2^n$ .

## Problem Set 1.2

1. (a)  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$  (b)  $\{1, 2, 3, 4\}$   
 (c)  $\{0, 1, 2, 3, 4\}$  (d)  $C$  (e)  $\{0\}$  (f)  $\phi$   
 (g)  $\{x \in R \mid x < 10, x \neq 0, 1, 2, 3, 4\}$   
 (h)  $\{(0, 1), (0, 2), \dots, (0, 10), (1, 1), (1, 2), \dots, (1, 10), (2, 1), (2, 2), \dots, (2, 10), (3, 1), (3, 2), \dots, (3, 10), (4, 1), (4, 2), \dots, (4, 10)\}$ .
2. (a)  $\{0, 1, 2, \dots, 20\}$  (b)  $\{0, 1, 2, 3, 4, 7, 14, 21, 28, \dots\}$   
 (c)  $\{x \in N \mid 3 < x < 20\}$  (d)  $\{7, 14\}$  (e)  $\{4\}$  (f)  $\phi$  (g)  $\{0\}$   
 (h)  $A$  (i)  $\{1, 2, 3\}$  (j)  $\{x \in N \mid x < 20, x \neq 7, 14\}$   
 (k)  $\{x \in N \mid x > 3, x \text{ not divisible by } 7\}$   
 (l)  $\{(0, 1), (0, 2), \dots, (0, 20), (1, 1), (1, 2), \dots, (1, 20), (2, 1), (2, 2), \dots, (2, 20), (3, 1), (3, 2), \dots, (3, 20), (4, 1), (4, 2), \dots, (4, 20)\}$ .
4. (a)  $\{(\alpha, \beta), (\alpha, \delta), (\alpha, \theta), (\beta, \beta), (\beta, \delta), (\beta, \theta), (\gamma, \beta), (\gamma, \delta), (\gamma, \theta)\}$   
 (b)  $\{(\alpha, \alpha), (\alpha, \gamma), (\alpha, \epsilon), (\beta, \alpha), (\beta, \gamma), (\beta, \epsilon), (\gamma, \alpha), (\gamma, \gamma), (\gamma, \epsilon)\}$   
 (c)  $\{(\beta, \alpha), (\beta, \gamma), (\beta, \epsilon), (\delta, \alpha), (\delta, \gamma), (\delta, \epsilon), (\theta, \alpha), (\theta, \gamma), (\theta, \epsilon)\}$ .
5. (a)  $\{x \mid x > 7 \text{ or } x < 0\}$  (b)  $\{x \mid x \neq 1\}$  (c)  $R$

(d)  $\{x \mid -1 < x < 1\}$  (e)  $\emptyset$  (f)  $\{x \mid x > 1\}$ .

7.  $A = \{1\}$ ,  $B = \{1\}$ ,  $C = \{2\}$ ,  $D = \{3\}$ .

### Problem Set 1.3

1. (a) Reflexive (b) None  
 (c) Reflexive, symmetric, transitive, and hence an equivalence relation (d) Transitive (e) Reflexive and transitive  
 (f) Symmetric (g) Transitive (h) Symmetric  
 (i) Reflexive, symmetric, transitive, and hence an equivalence relation  
 (j) Reflexive, symmetric, transitive, and hence an equivalence relation  
 (k) Symmetric and transitive.

### Problem Set 1.4

1. (b) and (c) are functions.
2. (a) (i) Not possible (ii)  $f(a) = f(b) = f(c) \Rightarrow f(d) = 1$   
 (iii) Not possible (iv)  $f(a) = 0, f(b) = 1, f(c) = 2, f(d) = 3$   
 (b) (i) Not possible (ii)  $f(1) = 0, f(2) = f(3) = 1$   
 (iii)  $f(1) = 0, f(2) = 2, f(3) = 3$  (iv) Not possible  
 (c) (i)  $f(0) = f(1) = 1, f(2) = 2, f(3) = 3$   
 (ii)  $f(0) = f(1) = 2, f(2) = f(3) = 3$   
 (iii) Not possible (iv) Not possible  
 (d) (i)  $f(x) = \tan x$  (ii)  $f(x) = \sin x$   
 (iii)  $f(x) = e^x$  (iv)  $f(x) = 3x + 4$ .
3. (a)  $(-\infty, -1] \cup [1, \infty)$  (b)  $[-2, 2]$  (c)  $\{x \in \mathbb{R} \mid x \neq -1\}$   
 (d)  $\mathbb{R}$  (e)  $\mathbb{R}$  (f)  $\{0\} \cup [1, \infty)$ .
4. (a)  $C$  (b)  $\{z \in C \mid z \neq 0\}$  (c)  $C$  (d)  $C$  (e)  $C$  (f)  $C$ .
5. 4(a)  $[0, \infty)$  4(b)  $[0, 2]$  4(c)  $\{x \in \mathbb{R} \mid x \neq 1\}$  4(d)  $[6, \infty)$   
 4(e)  $[-1/2, 1/2]$  4(f)  $[0, \infty)$   
 5(a)  $\{z \in C \mid \operatorname{Im} z = 0, \operatorname{Re} z \geq 0\}$  5(b)  $\{z \in C \mid |z| = 1\}$   
 5(c)  $\{z \in C \mid \operatorname{Im} z = 0\}$  5(d)  $\{z \in C \mid |z| = 1\}$   
 5(e)  $\{z \in C \mid \operatorname{Im} z = 0, \operatorname{Re} z > 0\}$  5(f)  $C$ .
6. (c) is one-one. None is onto.
8. (a), (c), and (f) represent functions.

### Problem Set 1.5

3. (a) In  $\mathbb{R}$  define  $a * b = a$   
 (b) In  $\mathbb{R} \times \mathbb{R}$  define  $(x_1, x_2) * (y_1, y_2) = (\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2})$   
 (c) In  $\mathbb{R}$  define  $a * b = a - b + ab$ .

**Problem Set 1.6**

6. Additive identity, additive inverses, and multiplicative inverses are not in  $N$ .

**Problem Set 1.7**

2.  $g \circ f: x \mapsto \sin^2 x$ ,  $-\pi < x < \pi$ .  $f \circ g$  is not defined since  $R(g) \not\subseteq D(f)$ . If  $D(g) = [-\sqrt{\pi}, \sqrt{\pi}]$ , then  $f \circ g$  is defined and  $f \circ g: x \mapsto \sin x^2$ ,  $-\sqrt{\pi} < x < \sqrt{\pi}$ .
3.  $g \circ f: x \mapsto \sqrt{1+x^2}$ ,  $-1 < x < 1$ .  $f \circ g$  is not defined since  $R(g) \not\subseteq D(f)$ . If  $D(g) = [0, 1]$ , then  $f \circ g$  is defined and  $f \circ g: x \mapsto 1+x$ ,  $0 < x < 1$ .
6.  $(f+g)(x) = \sin x + x^2$ ;  $(f-g)(x) = \sin x - x^2$ ;  
 $(fg)(x) = x^2 \sin x$ .
7.  $(f+g)(x) = 1 + x^2 + \sqrt{x}$ ;  $(f-g)(x) = 1 + x^2 - \sqrt{x}$ ;  
 $(fg)(x) = (1+x^2)\sqrt{x}$ .
10. No.

**Problem Set 2.1**

2. (a) A plane parallel to the  $xz$ -plane  
(b) A plane parallel to the  $xy$ -plane  
(c) A line passing through  $(1, 0, 0)$  and parallel to the line  $y = 2z$  in the  $yz$ -plane  
(d) A parabola in the plane  $y = -5$ ; the axis of the parabola is a line parallel to the  $x$ -axis and passing through  $(0, -5, 0)$   
(e) The projection of the rectangular hyperbolas  $yz = 1$  and  $yz = -1$  (lying in the  $yz$ -plane) on the plane through  $(2, 0, 0)$  parallel to the  $yz$ -plane  
(f) The half space on that side of the  $yz$ -plane in which  $x < 0$ , and the  $yz$ -plane  
(g)  $x = y$  is the plane perpendicular to the  $xy$ -plane and containing the line  $y = x$  in the  $xy$ -plane; the required set of points is the union of the plane  $x = y$  and the half space on that side of this plane that contains the positive  $x$ -axis  
(h) The projection of the line  $y = 2z$  (lying in the  $yz$ -plane) on the plane through  $(3, 0, 0)$  parallel to the  $yz$ -plane along with the half plane determined by this projection and containing the point  $(3, 1, 0)$ .
3. (a)  $\sqrt{3}$  (b)  $\sqrt{14}$  (c)  $\sqrt{10}$  (d) 3.
4.  $(1, 2, 3), (-1, 2, 3), (-1, 11/2, 3), (1, 11/2, 3), (1, 2, 9/2), (-1, 2, 9/2), (-1, 11/2, 9/2), (1, 11/2, 9/2); \sqrt{37}/\sqrt{2}$ .

**Problem Set 2.2**

- $\sqrt{13}$ ;  $\tan \theta = 3/2$ ,  $0 < \theta < \pi/2$
  - $\sqrt{10}$ ;  $\tan \theta = -1/3$ ,  $3\pi/2 < \theta < 2\pi$
  - $3\sqrt{5}$ ;  $\tan \theta = 2$ ,  $0 < \theta < \pi/2$
  - $\sqrt{5}$ ;  $\tan \theta = -2$ ,  $\pi/2 < \theta < \pi$
  - $\sqrt{2}$ ;  $\theta = \pi/4$
  - $\sqrt{13}$ ;  $\tan \theta = -3/2$ ,  $3\pi/2 < \theta < 2\pi$
  - $\sqrt{17}$ ;  $\tan \theta = -4$ ,  $\pi/2 < \theta < \pi$
  - $2\sqrt{5}$ ;  $\tan \theta = 1/2$ ,  $\pi < \theta < 3\pi/2$ .
- $\sqrt{14}$
  - 5
  - $\sqrt{14}$
  - $\sqrt{21}$ .
- (6, -9)
  - (4, -4, 12)
  - (-33, -9)
  - (2, 17, 12)
  - $13i + 18j - 11k$ .
- $2i + 3j$
  - $-i + 4j$
  - $-3i - 5j$
  - $5i + 3j - 2k$
  - $2i + k$
  - $-i + 2j$ .
- $-u + 2v$
  - $3u - 20v + 12w$
  - $-u - 11v + 7w$
  - $3u + 2v - 2w$
  - $4u + 12v - 9w$
  - $-2u - 25v + 15w$ .
- $(2/\sqrt{13}, 3/\sqrt{13})$
    - $(3/\sqrt{10}, -1/\sqrt{10})$
    - $(1/\sqrt{5}, 2/\sqrt{5})$
    - $(-1/\sqrt{5}, 2/\sqrt{5})$
    - $(i + j)/\sqrt{2}$
    - $(2i - 3j)/\sqrt{13}$
    - $(-i + 4j)/\sqrt{17}$
    - $(-2i - j)/\sqrt{5}$
  - $(2/\sqrt{14}, -1/\sqrt{14}, 3/\sqrt{14})$
    - $(3/5, 0, 4/5)$
    - $(3i + 2j - k)/\sqrt{14}$
    - $(-i - 2j + 4k)/\sqrt{21}$ .
- $(-1/\sqrt{2}, -1/\sqrt{2})$
  - $(\sqrt{3}/2, 1/2)$
  - $(1/\sqrt{2}, -1/\sqrt{2})$ .
- (1, -1, 2)
  - (2, 3, 1)
  - (4, -2, 1)
  - (3, 2, 6)
  - (2, -4)
  - (6, 2).
- $A(1/2, 1/2)$ ,  $B(3/2, 7/2)$
  - $A(1/2, -1)$ ,  $B(-1/2, 1)$
  - $A(3/2, -1/2, 1)$ ,  $B(9/2, -3/2, 3)$
  - $A(3, 3, -5/2)$ ,  $B(1, 3, -3/2)$ .

**Problem Set 2.3**

- 2
  - 32
  - 4
  - 2
  - 5
  - 4
  - 4
  - 0.
- $1/\sqrt{5}$
  - $-32/\sqrt{1073}$
  - $-4/\sqrt{65}$
  - $-1/\sqrt{442}$
  - $5/\sqrt{385}$
  - $-4/3\sqrt{14}$
  - $-4/\sqrt{42}$
  - 0.
- $2/\sqrt{10}$
  - $-32/\sqrt{37}$
  - $-4/\sqrt{13}$
  - $1/\sqrt{13}$
  - $5/\sqrt{11}$
  - $-4/\sqrt{6}$
  - $-4/\sqrt{14}$
  - 0.
- (1, 1)
  - $(64/29, -160/29)$
  - $(-4i - 8j)/5$
  - $(-3i + 5j)/17$
  - $(1/7, 3/7, 5/7)$
  - $(-8/21, 4/21, -16/21)$
  - $(4i - 4j - 4k)/3$
  - 0.
- 3
  - 0
  - any real number
  - 2, 1.

**Problem Set 2.4**

3.  $\cos \alpha = -2/\sqrt{17}$ ,  $\cos \beta = 3/\sqrt{17}$ ,  $\cos \gamma = 2/\sqrt{17}$ .
4.  $\mathbf{r} = \mathbf{u} + t(\mathbf{v} - \mathbf{u})$ , where  $\mathbf{u}$  and  $\mathbf{v}$  are the position vectors of the given points.
7.  $\cos \theta = \frac{l'l'' + mm' + nn'}{\sqrt{(l^2 + m^2 + n^2)} \sqrt{(l'^2 + m'^2 + n'^2)}}$ .
8.  $\cos^{-1}(2/3)$ ,  $\cos^{-1}(-2/3)$ ,  $\cos^{-1}(1/3)$ .
9.  $2x + 3y + 6z = 35$ .
10.  $3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ .
11. (a)  $\pi/4$  (b)  $\pi/2$ .

**Problem Set 3.1**

1. (a)  $(1, 5, -1, 12)$  (b)  $(16, -100, 2, -6)$  (c)  $(0, 0, 0, 0)$   
(d)  $(-3, 19, -12, 13)$  (e)  $(0, 6a - 2b, 3a + 3b, 3a - 5b)$ .
2. Yes.
4. (a), (d), and (f) are vector spaces.
6. (a), (d), and (f) are vector spaces.
7. (a), (c), (d), and (f) are vector spaces.

**Problem Set 3.2**

4. (c), (g), and (i) are subspaces of  $V_3$ .
5. (b), (d), and (e) are subspaces of  $\mathcal{P}$ .
6. (a), (b), (d), (e), and (f) are subspaces of  $\mathcal{C}$  (a, b).

**Problem Set 3.3**

1. (a), (c), and (d) are in  $[S]$ . 2. (c) is in  $[S]$ .
6. (a)  $xy$ -plane (b)  $xy$ -plane (c)  $V_2$  (d)  $V_3$ .

**Problem Set 3.4**

5. (a)  $\{(x_1, x_2) \in V_2 \mid x_1 = 0\}$ ; subspace of  $V_2$   
(b)  $\{f \in \mathcal{C}(-2, 2) \mid f(1) = f(-1) = 0\}$ ; subspace of  $\mathcal{C}(-2, 2)$   
(c)  $\{f \in \mathcal{C}(-2, 2) \mid \lim_{x \rightarrow 1} f(x) = 0 \text{ and } \lim_{x \rightarrow 2} f(x) = 1\}$ ; not a subspace  
(d)  $\{p \in \mathcal{P} \mid p(x) = p(-x)\}$ ; subspace of  $\mathcal{P}$ .
6. (a)  $\{(2, 2), (4, 1), (1, 1), (3, 0)\}$ ; subset  
(b)  $\{(4, 3), (8, 6), (3/2, 1), (11/2, 4), (1 + \sqrt{2}, \pi - 2), (5 + \sqrt{2}, \pi + 1)\}$ ; subset  
(c)  $\{(5/2 - t, 11/3 - 2t) \mid 0 < t < 1\}$ ; subset



(It is the line segment joining the points  $(3/2, 5/3)$  and  $(5/2, 11/3)$ .)

- (d)  $\{(2 + 3t, 3 + 4t) \mid 1 < t < 2\}$ ; subset  
(It is the line segment joining the points  $(5, 7)$  and  $(8, 11)$ .)
- (e)  $\{(3 - t, 7 + 2t) \mid 0 < t < 1\}$ ; subset  
(It is the line segment joining the points  $(3, 7)$  and  $(2, 9)$ .)
- (f)  $\{(x, 2) \mid x \geq 1/2\}$ ; subset
- (g)  $\{(x, y) \mid 2x + 3y = 17\}$ ; parallel
- (h)  $\{(x, y) \mid (x - 1)^2 + (y - 5)^2 = 1\}$ ; subset  
(It is the unit circle centred at  $(1, 5)$ .)
- (i)  $\{(3t_1 + 2t_2, 4t_1 + 5t_2) \mid 0 < t_1 < 1, 1 < t_2 < 2\}$ ; subset  
(It is the interior of the parallelogram  $PQRS$ , where the points  $P, Q, R$ , and  $S$  are  $(2, 5), (5, 9), (7, 14)$ , and  $(4, 10)$ , respectively, including the edges  $PS$  and  $QR$ .)
- (j)  $\{(t_1, t_2) \mid 0 < t_1 < 1, 2 < t_2 < 4\}$ ; subset  
(It is the rectangle with vertices  $(0, 2), (0, 4), (1, 4), (1, 2)$ , and its interior.)
- (k)  $V_2$ ; subspace (l)  $V_2$ ; subspace.
7. (a)  $\{(1 + t, 2 + 2t, 1) \mid t \text{ a scalar}\}$ ; parallel  
(It is the line through the point  $(1, 2, 1)$  and parallel to the vector  $(1, 2, 0)$ .)
- (b)  $\{(x, y, z) \mid x + y + z = 3\}$ ; parallel  
(It is the plane through the point  $(3, 1, -1)$  and parallel to the plane  $x + y + z = 0$ .)
- (c)  $\{(1 + \alpha, -3 + 2\alpha, 4 + 3\alpha + \beta) \mid \alpha, \beta \text{ scalars}\}$ ; parallel  
(It is the plane through the point  $(1, -3, 4)$  and parallel to the plane  $2x - y = 0$ .)
- (d)  $\{(\alpha + 3\beta, 2\alpha + \beta, 3\alpha) \mid \alpha, \beta \text{ scalars}\}$ ; subspace
- (e)  $V_3$ ; subspace (f)  $V_3$ ; subspace (g)  $B$ ; subspace.

12. (b) Base space :  $\{(x_1, x_2, \dots, x_n) \mid \sum_{i=1}^n \alpha_i x_i = 0\}$

Leader :  $\frac{\beta}{\alpha_i} e_i, \alpha_i \neq 0$ .

13. (a) Base space : set of all constant functions Leader :  $x^3$
- (b) Base space : set of all constant functions Leader :  $x^{3/2}$
- (c) Base space :  $\{\alpha x + \beta \mid \alpha, \beta \text{ scalars}\}$  Leader :  $-\sin x$
- (d) Base space :  $\{\alpha x + \beta \mid \alpha, \beta \text{ scalars}\}$  Leader :  $x^4/12 + x^3/3 + 3x^2/2$
- (e) Base space :  $\{f \in \mathcal{C}(0, 2) \mid f(1) = 0\}$  Leader :  $x + 1$ .
14. (a)  $U \cap W = \{0\}$  (b)  $U + W = \{f_1 + f_2 \mid f_1, f_2 \in \mathcal{C}'(-a, a), f_1 \text{ is odd and } f_2 \text{ is even}\}$ ;  $U + W = \mathcal{C}'(-a, a) = U \oplus W$ .

**Problem Set 3.5**

1. (a), (b), and (d) are LI.
2. (a), (c), and (e) are LD.
3. (a) and (c) are LI.
4. (a) and (e) are LI
11. (a)  $S$  (b)  $S$  (c)  $\{(1, 1, 2), (-3, 1, 0), (1, -1, 1)\}$  (d)  $S$   
(e)  $\{(1/2, 1/3, 1), (2, 3/4, -1/3)\}$ .
12. (a)  $\{(1, 0, 0, 0), (1, 1, 0, 0), (1, 1, 1, 1)\}$  (b)  $S$   
(c)  $\{(1, 1, 1, 0), (3, 2, 2, 1), (1, 1, 3, -2), (1, 1, 2, 1)\}$   
(d)  $S$  (e)  $\{(1, 2, 3, 0), (-1, 7, 3, 3)\}$ .
13. (a)  $S$  (b)  $\{1, x + x^2, x - x^2\}$   
(c)  $S$  (d)  $\{x^2 - 4, x + 2, x - 2\}$ .
14. (a)  $S$  (b)  $\{\sin^2 x, \cos 2x\}$   
(c)  $\{\sin x, \cos x\}$  (d)  $\{\ln x\}$  (e)  $S$ .
15. 1(c)  $(1, 2, -3)$  1(e)  $(0, 0, 0)$  2(a)  $(0, 0, 1, 1)$   
2(c)  $(1, 2, 6, -5)$  2(e)  $(1, -1, 1, -1)$  3(b)  $3x$  3(d)  $x^2/3$   
4(b)  $1$  4(c)  $\sin(x + 1)$  4(d)  $\ln x^2$ .

**Problem Set 3.6**

1. (a) Not a basis ;  $\{(1, 2, 3), (3, 1, 0)\}$   
(b) Not a basis ;  $\{(1, 1, 1), (1, 2, 3)\}$   
(c) Not a basis ;  $\{(0, 0, 1), (1, 0, 1), (1, -1, 1)\}$   
(d) Basis (e) Not a basis ;  $S$ .
2. (a) Basis (b) Basis (c) Not a basis (d) Basis  
(e) Basis (f) Basis (g) Not a basis (h) Basis.
3. (a) 2 (b) 2 (c) 3 (d) 3 (e) 2.
4. (a) 4 (b) 4 (c) 2 (d) 4 (e) 5 (f) 4 (g) 3 (h) 3.
5.  $\{(3, -1, 2), (1, 0, 0), (1, 1, 0)\}, \{(3, -1, 2), (0, 1, 1), (2, 1, 2)\}$ .
6. (a) 1 ;  $\{(3, 2, 1)\}$  (b) 3 ;  $\{(1, 2, 3), (0, 1, 2), (1, -2, 3)\}$ .
8. (a)  $\{1, x\}$  (b)  $\{(x - x_0)^2, (x - x_0)^3, (x - x_0)^4, (x - x_0)^5\}$   
(c) See Problem 11  
(d)  $\{(1, 0, -1, 3, 0), (0, 1, -1, 0, 0), (0, 0, 0, 7, 1)\}$   
(e)  $\{(x - x_0), (x - x_0)^2, (x - x_0)^3, (x - x_0)^4\}$ .
12. (a)  $-1/2, -1/2, 3/2$  (b)  $-3/2, -5/2, 7/2$   
(c)  $\frac{1}{2}x_1 - x_3, \frac{1}{2}x_1 - x_3 + x_3, -\frac{1}{2}x_1 + x_3$   
(d)  $-\frac{1}{\sqrt{2}} - e, -\frac{1}{\sqrt{2}} - \pi + e, \frac{1}{\sqrt{2}} + \pi$   
(e)  $-21/4, 13/2, 47/12$  (f)  $2, 0, -1$ .
13. (a)  $-1, 6, -2$  (b)  $-2, -1, 3$  (c)  $-1, 1, -1$ .

14.  $\{(1, 0, 0, 0), (-5, 3, 4, -6), (0, 0, 1, 0), (0, 0, 0, 1)\}$ .
15.  $\sum_{i=1}^n \alpha_{1i} x_i, \sum_{i=1}^n \alpha_{2i} x_i, \dots, \sum_{i=1}^n \alpha_{ni} x_i.$
16. 2.
17.  $A = [(1, 0, 0, 0), (0, 1, 0, 0)], B = [(1, 1, 5, 2), (1, 2, 3, 0), (1, 1, 1, 1)].$

### Problem Set 4.1

2. (a), (d), (f), (g), (h), (j), (k), (l), (n), (o), (p), and (q) are linear.
3. (a) Yes ;  $T(x, y) = (-x/3 + 5y/3, 4x/3 - 2y/3)$   
 (b) Yes ;  $T(x, y) = (2y, y)$  (c) No  
 (d) Yes ;  $T(x, y, z) = (y + z, x/2 + 2y - z/2, 2x - 2y + 2z)$   
 (e) Yes ;  $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = (-\alpha_0 + 2\alpha_1) + \alpha_1 x + 3(\alpha_0 - \alpha_1)x^2$   
 (f) Yes ;  $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4) = -2\alpha_0 + 3\alpha_1 + 4\alpha_2$   
 (g) Yes ;  $T(x, y) = (x - iy, x - y - iy).$
4.  $T(x, y) = (0, y - x).$
5.  $T(x, y) = (\frac{3x+y}{2}, y).$

### Problem Set 4.2

1. (a)  $V_2$ ; 2 (b)  $[(1, 1, 0), (0, 1, 1)]$ ; 2 (c)  $V_3$ ; 3 (d)  $V_3$ ; 3  
 (e)  $V_3$ ; 3 (f)  $[(1, 1, 1, 0), (0, 1, 1, 0), (0, 0, 1, 1)]$ ; 3  
 (g)  $V_4$ ; 4 (h)  $\{p \in \mathcal{P} \mid p(0) = 0\}$  (i)  $\{p \in \mathcal{P} \mid p(0) = 0\}$   
 (j)  $\mathcal{P}$  (k)  $\mathcal{C}'(0, 1)$  (l)  $\mathcal{C}(0, 1).$
2. (a)  $V_0$ ; 0 (b)  $V_0$ ; 0 (c)  $V_0$ ; 0 (d)  $V_0$ ; 0  
 (e)  $[(1, -1, 1, 1)]$ ; 1 (f)  $V_0$ ; 0 (g)  $V_0$ ; 0 (h)  $V_0$ ; 0  
 (i) Set of all constant polynomials; 1 (j)  $V_0$ ; 0  
 (k)  $V_0$ ; 0 (l) Set of all constant functions; 1.
6.  $T(x_1, x_2, x_3) = (4x_1 - 3x_2 + x_3, 8x_1 - 6x_2 + 2x_3, 0).$
7.  $T(x_1, x_2, x_3) = (x_1 + x_3, x_2 + 2x_3, -4x_1 + 3x_2 + 2x_3).$
8. (a) : (a), (b), (c), (d), (f), (g), (h), (j), and (k) are one-one  
 (b) : (a), (c), (d), (e), (g), (j), (k), and (l) are onto  
 (c) : (a), (c), (d), (g), (j), and (k) are one-one and onto.

### Problem Set 4.4

1. All are nonsingular;  
 $R^{-1}(x_1, x_2, x_3) = \frac{1}{2}(4x_1 + x_2 - 3x_3, 4x_1 - 4x_2 - 3x_3, -x_1 + x_2 + 2x_3)$

$$S^{-1}(x_1, x_2, x_3) = \frac{1}{2}(7x_1 + x_2, 7x_1 + 7x_2 + 2x_3, -x_3)$$

$$T^{-1}(x_1, x_2, x_3) = \frac{1}{6}(-9x_2, -2x_1 - 5x_2 - 3x_3, 5x_1 + 8x_2 + 3x_3).$$

2. (a)  $T^{-1}(x_1, x_2) = (x_1/\alpha_1, x_2/\alpha_2)$

(b)  $T^{-1}(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3)$

(c)  $T^{-1}(\sigma_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0/3 - \alpha_1 + 2\alpha_2/3) + (2\alpha_0/3 + \alpha_1 - 2\alpha_2/3)x + (-\alpha_0/3 + \alpha_2/3)x^2.$

### Problem Set 4.6

1. (a)  $(2S + 3T) : (x_1, x_2) \mapsto (7x_1 + 3x_2, 6x_1 + 8x_2)$

(b)  $(3S - 7T) : (x_1, x_2) \mapsto (-x_1 - 7x_2, 9x_1 + 12x_2).$

2. (a)  $(S + T) : (x_1, x_2, x_3) \mapsto (2x_1 - 2x_2 + x_3, 5x_1 + 6x_2 + x_3, -x_1 + 2x_3)$

(b)  $(3S - 2T) : (x_1, x_2, x_3) \mapsto (-4x_1 + 9x_2 + 3x_3, -5x_1 - 12x_2 + 3x_3, -3x_1 + x_3)$

(c)  $(\alpha S) : (x_1, x_2, x_3) \mapsto (\alpha x_2 + \alpha x_3, \alpha x_1 + \alpha x_3, -\alpha x_1 + \alpha x_3).$

3.

	$e_1$	$e_2$	$e_3$
(a) $R + 2S$	$3e_1 - e_2$	$e_1 + e_2 + e_3$	$5e_1 + 2e_2 - 10e_3$
(b) $2R + 5T$	$7e_1 - 3e_2 + 5e_3$	$17e_1 - 2e_2 - 23e_3$	$21e_1 - 2e_3$
(c) $S - T$	$-e_3$	$-3e_1 + e_2 + 5e_3$	$-2e_1 + e_2 - 5e_3$
(d) $R + S + 2T$	$4e_1 - 2e_2 + 2e_3$	$7e_1 - 9e_3$	$10e_1 + e_2 - 7e_3$
(e) $\alpha R + \beta S + \gamma T$	$(\alpha + \beta + \gamma)e_1 + (\alpha - \beta - \gamma)e_2 + \gamma e_3$	$(\alpha + 3\gamma)e_1 + (-\alpha + \beta)e_2 + (\alpha - 5\gamma)e_3$	$(3\alpha + \beta + 3\gamma)e_1 + \beta e_2 + (4\alpha - 7\beta - 2\gamma)e_3$

7. (a)  $T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2, -x_1 + x_3 - x_4, -x_2 + x_3, -x_3 + x_4),$

$$S(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3, x_1 + x_2 + x_3 + x_4, x_2 + x_4, 2x_3)$$

(b)  $T(x_1, x_2, x_3, x_4) = (x_2 + x_4, x_1 + x_3 + x_4, -x_2 + x_3, x_3),$

$$S(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 - 4x_3, 2x_2 + 2x_4, x_1 + x_3 - x_4, x_2 + 2x_3)$$

(c)  $T(x_1, x_2, x_3, x_4) = (x_2, x_2 + x_3 - x_4, 2x_1 + x_3, x_3 + 6x_4),$

$$S(x_1, x_2, x_3, x_4) = (x_1 + x_3/2 - 3x_4/2, -x_2/2 - x_3/2 + x_4/2, x_2/2 + 3x_4/2, 2x_1 + x_2 + x_3/2)$$

(d)  $T = S.$

## Problem Set 4.7

1.  $(ST)(x_1, x_2, x_3) = (x_1, x_1 + x_2 + x_3).$
2. (a)  $(ST)(x_1, x_2) = (2x_1 + 2x_2, 3x_1 + 3x_2)$   
 (b)  $(TS)(x_1, x_2) = (5x_1 + 4x_2, 0)$   
 (c)  $S^2(x_1, x_2) = (4x_1, 18x_1 + 16x_2)$   
 (d)  $(T^2S)(x_1, x_2) = (5x_1 + 4x_2, 0).$
3. (a)  $(ST)(x_1, x_2, x_3) = (4x_1 + 6x_2 + x_3, 2x_1 - 3x_2 + x_3, -2x_1 + 3x_2 + x_3)$   
 (b)  $(TS)(x_1, x_2, x_3) = (-3x_1 + 2x_2 - x_3, 6x_1 + 4x_2 + 10x_3, -x_1 + x_3)$   
 (c)  $(STS)(x_1, x_2, x_3) = (5x_1 + 4x_2 + 11x_3, -4x_1 + 2x_2, 2x_1 - 2x_2 + 2x_3)$   
 (d)  $(TST)(x_1, x_2, x_3) = (2x_1 + 21x_2 - x_3, 28x_1 + 6x_2 + 10x_3, -2x_1 + 3x_2 + x_3).$

4.

	$e_1$	$e_2$	$e_3$
(a) $ST$	$2e_1 - e_2 - 7e_3$	$-2e_1 - 8e_2 + 35e_3$	$e_1 - 5e_2 + 14e_3$
(b) $RT$	$3e_1 + 2e_2 + 3e_3$	$-12e_1 + 3e_2 - 20e_3$	$-3e_1 + 3e_2 - 8e_3$
(c) $RST$	$-20e_1 + 3e_2 - 29e_3$	$95e_1 + 6e_2 + 132e_3$	$38e_1 + 6e_2 + 51e_3$
(d) $R(S + T)$	$3e_1 + 4e_2 + 2e_3$	$-11e_1 + 2e_2 - 19e_3$	$-22e_1 + 3e_2 - 35e_3$
(e) $T^2$	$e_1 - e_2 + 4e_3$	$-12e_1 - 3e_2 + 13e_3$	$-3e_1 - 3e_2 + 7e_3$
(f) $T^2ST$	$35e_1 + 22e_2 - 54e_3$	$-11e_1 - 79e_2 + 133e_3$	$19e_1 - 28e_2 + 37e_3$

5.  $S(x_1, x_2) = (x_1 - x_2, x_2 - x_1), T(x_1, x_2) = (x_2, x_1).$
9. (a), (b), (c) Range :  $V_2$  ; kernel :  $V_0$  ; rank : 3 ; nullity : 0.
14. (b)  $S_1S_2$  is idempotent if  $S_1S_2 = S_2S_1$  ;  $S_1 + S_2$  is idempotent if  $S_1S_2 + S_2S_1 = 0$   
 (d)  $T(x_1, x_2, x_3) = (x_1, x_2, 0), S(x_1, x_2, x_3) = (0, 0, x_3).$
15. (b) Yes ; 5 (c) Yes (d) No

(e) Less than or equal to the minimum of the degrees of nilpotence of  $S$  and  $T$ ;  $S + T$  is nilpotent.

$$17. (b) (I + \lambda T)^{-1}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3) = \alpha_0 + \alpha_1 x + (\alpha_2 - \lambda \alpha_0 - 2\lambda \alpha_1)x^2 + (\alpha_3 - \lambda \alpha_1 - \lambda \alpha_2 + \lambda^2 \alpha_0 + 2\lambda^2 \alpha_1)x^3.$$

### Problem Set 4.8

1.

	Range	Kernel	Pre-image of $(3, -1, 2)$
$R$	$V_3$	$V_0$	$(1, 2, 0)$
$S$	$V_3$	$V_0$	$(23/7, 18/7, -2/7)$
$T$	$V_3$	$V_0$	$(1, -7/9, 13/9)$

2. (b)  $\phi$  (c)  $(8/7, -18/7, 3)$  (d)  $(1, 3, 2)$

(e)  $(1, -1, 3, 0) + [(1, -1, 1, 1)]$ .

3. (a): (h) 1 (i)  $x + [1]$  (j)  $-x/2$  (k)  $x \operatorname{cosec} x$   
(l)  $-(1+x)e^{-x} + [1]$

(b): (h)  $x$  (i)  $(x^2/2) + [1]$  (j)  $-(1+x^2)/2$   
(k)  $x^2 \operatorname{cosec} x$  (l)  $-(2+2x+x^2)e^{-x} + [1]$ .

### Problem Set 4.9

1. (a)  $2y = (C + e^{x^2})e^x$

(b)  $y = \frac{C}{x} + \frac{x^2}{3} + 2 \cos x + (x - \frac{2}{x}) \sin x, x \neq 0; y = 0$   
when  $x = 0$

(c)  $3y = C(3x-4)^{-5/3} - x^2(3x-4)^{-5/3}, x \neq 4/3$

(d)  $y = Ce^{-\alpha x} + e^{\beta x}/(\alpha + \beta), \alpha + \beta \neq 0; y = (C+x)e^{-\alpha x}$   
if  $\beta = -\alpha$  (e)  $x = Cy + y^3$

(f)  $y = Ce^{\beta x} - \frac{\alpha x}{\beta} - \frac{\alpha + \beta \delta}{\beta^2}, \beta \neq 0; y = \frac{\alpha x^2}{2} + \delta x + C$

if  $\beta = 0$  (g)  $x = Ce^{-\tan^{-1} y} + \tan^{-1} y - 1$

(h)  $x = Ce^{\sin y} + e^{\sin y} \int e^{-\sin y} \cos^2 y dy$

(i)  $(y-1)x = (y+1)(C-y+2 \ln(y+1)), y > 1$

(j)  $x = C \cos y + \sin y$  (k)  $y(1+x^2) = C - \cos x$   
+  $(x-1)e^x + x^2/3$

(l)  $y\sqrt{1+x^2} = C + \frac{1}{2} \ln \left( \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} \right)$ .

2. (a)  $y(C + \sinh^{-1} x) = \sqrt{1+x^2}$

(b)  $y^2(Ce^{x^2} + 1 + x^2) = 1$

(c)  $y^2(C + 2x) = e^{x^2}$

(d)  $x(C - y^2) = e^y$  (e)  $x(C \cos y + \sin y) = 1$ .

3. (a) LI; yes (b) LI; no (c) LD; no (d) LD; no

(e) LD; no (f) LD; no (g) LI; yes (h) LI; yes.

**Problem Set 5.1**

1. (a)  $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$  (b)  $\begin{bmatrix} -\frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$ .

2. (a)  $\begin{bmatrix} 2 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} -\frac{7}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 4 & 23 \end{bmatrix}$ .

3.  $\begin{bmatrix} 4 & 2 & 3 & 1 \\ -1 & 0 & 1 & 0 \\ 4 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 \\ 13 & 2 & 2 & 5 \end{bmatrix}$ .

4.  $\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & n \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$ .

5. (a)  $\begin{bmatrix} 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$  (b)  $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$ .

6. (a)  $\begin{bmatrix} -1 & -1 & -3 \\ 1 & 1 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$  (b)  $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ .

7.  $\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}$ .

8. (a)  $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$  (b)  $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$  (c)  $\begin{bmatrix} \frac{1}{2} & 1 & 1 \\ 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d)  $\begin{bmatrix} & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$  (e)  $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

(f)  $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$  (g)  $\begin{bmatrix} 3 & 2 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ .

### Problem Set 5.2

- (a)  $T: V_4 \rightarrow V_3$ ,  
 $T(x_1, x_2, x_3, x_4) = (x_1 + x_2 + 2x_3 + 3x_4, x_1 + x_3 - x_4, x_1 + 2x_2)$

(b)  $T: V_4 \rightarrow V_3$ ,  
 $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 + 11x_3 - 8x_4, 11x_1 + 7x_2 + 22x_3 - 19x_4, 13x_1 + 8x_2 + 30x_3 - 23x_4)$
- (a)  $T: V_3 \rightarrow V_3$ ,  $T(x_1, x_2, x_3) = (x_1, x_2, x_3)$

(b)  $T: V_3 \rightarrow V_3$ ,  $T(x_1, x_2, x_3) = (x_1 + 2x_2 - 2x_3, -x_1 + x_2 + 2x_3, x_1 + x_2 + x_3)$

(c)  $T: V_3 \rightarrow V_3$ ,  $T(x_1, x_2, x_3) = \frac{1}{6}(-x_1 - 4x_2 + 6x_3, 3x_1 + 3x_2, -2x_1 + x_2 + 3x_3)$
- (a)  $T: V_3 \rightarrow V_2$ ,  $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 3x_1 + x_2)$

(b)  $T: V_3 \rightarrow V_2$ ,  $T(x_1, x_2, x_3) = (2x_1 + 8x_2 - 6x_3, 2x_1 - 8x_2 + 4x_3)$

(c)  $T: V_3 \rightarrow V_2$ ,  $T(x_1, x_2, x_3) = (5x_1 - 2x_2, -x_1 - 2x_3)$
- (a)  $T: V_2 \rightarrow V_3$ ,  $T(x_1, x_2) = (x_1 + 2x_2, x_2, -x_1 + 3x_2)$

(b)  $T: V_3 \rightarrow V_3$ ,  $T(x_1, x_2) = (2x_2 - x_1, x_2, 3x_2 - 3x_1)$

(c)  $T: V_3 \rightarrow V_3$ ,  $T(x_1, x_2) = \frac{1}{2}(2x_1 + 4x_2, -x_1 - 2x_2, -17x_1 + x_2)$
- $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$ .



**Problem Set 5.3**

$$1. \quad (a) \begin{bmatrix} 9 & 4 & 13 \\ 12 & 9 & -5 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 6 & 7 \\ -7 & 1 & 5 \end{bmatrix}.$$

$$2. \quad (a) \begin{bmatrix} 8 & 6 \\ 19 & -2 \end{bmatrix} \quad (b) \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}.$$

$$3. \quad (a) \begin{bmatrix} -16 & 10 & -6 & 8 \\ -8 & -28 & -40 & -16 \\ -6 & -4 & 4 & -6 \end{bmatrix} \quad (b) \begin{bmatrix} 18 & 1 & 19 & 12 \\ 2 & 28 & 38 & 18 \\ 19 & 8 & 6 & 5 \end{bmatrix}$$

$$(c) \begin{bmatrix} 2 & -17 & -15 & -28 \\ 10 & 8 & 14 & 2 \\ -15 & -4 & -14 & 3 \end{bmatrix}.$$

$$4. \quad (a) \begin{bmatrix} 2 & -7 & -3 \\ 14 & -26 & -12 \\ 6 & 9 & 5 \end{bmatrix} \quad (b) \begin{bmatrix} \frac{7}{6} & -\frac{1}{2} & \frac{11}{6} \\ 1 & -\frac{5}{6} & \frac{1}{6} \\ \frac{7}{2} & \frac{5}{6} & -\frac{2}{3} \end{bmatrix}$$

$$(c) \begin{bmatrix} \frac{17}{18} & -\frac{1}{3} & \frac{22}{18} \\ \frac{2}{3} & -\frac{1}{6} & \frac{7}{18} \\ \frac{17}{6} & \frac{22}{18} & -\frac{4}{3} \end{bmatrix}.$$

$$6. \quad \begin{bmatrix} -1 & -5 & 0 \\ -1 & -2 & 3 \\ 0 & -9 & 5 \\ -1 & -4 & -1 \end{bmatrix}.$$

$$7. \quad \begin{bmatrix} -1 & -17 & -33 & 14 \\ 5 & 5 & 5 & -4 \\ 0 & 10 & 44 & -6 \\ 0 & 0 & -14 & -2 \\ 0 & 0 & 0 & 36 \end{bmatrix}.$$

$$9. \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$10. \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

11.  $\dim V = 5$ . The matrices with respect to the ordered basis  $\{\sin x, \cos x, \sin x \cos x, \sin^2 x, \cos^2 x\}$  are

$$(a) \begin{bmatrix} 3 & -2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & -4 \\ 0 & 0 & -2 & 3 & 0 \\ 0 & 0 & 2 & 0 & 3 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -8 & -2 & 2 \\ 0 & 0 & 1 & -2 & 6 \\ 0 & 0 & -1 & 6 & -2 \end{bmatrix}.$$

### Problem Set 5.4

1. (a)  $-1$  (b)  $9$  (c)  $3$ .

2. (a)  $(2, 3, 3)^T$  (b)  $(8, 19, 7)^T$  (c) The matrix  $[-1]$

$$(d) \begin{bmatrix} 3 & 1 & 2 \\ 6 & 2 & 4 \\ -3 & -1 & -2 \\ 3 & 1 & 2 \end{bmatrix}$$

$$(e) \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

$$3. (a) AB = \begin{bmatrix} 9 & 7 \\ 6 & 19 \end{bmatrix}, BA = \begin{bmatrix} 2 & 1 & 3 \\ 7 & 11 & 6 \\ 11 & 8 & 15 \end{bmatrix}$$

$$(b) BA = \begin{bmatrix} 5 & 5 & 6 \\ -2 & -1 & -3 \\ 9 & -8 & 21 \\ 11 & 13 & 12 \end{bmatrix}$$

$$(c) AB = \begin{bmatrix} 22 & 56 & 13 \\ 3 & 5 & 7 \\ 3 & 9 & 4 \end{bmatrix}, BA = \begin{bmatrix} 7 & 9 & 10 \\ 3 & 3 & 2 \\ 24 & 8 & 21 \end{bmatrix}$$

$$(d) AB = \begin{bmatrix} 3 & 11 & 24 & 12 & 20 \\ 7 & 15 & 24 & 28 & 36 \end{bmatrix}$$

$$(e) BA = \begin{bmatrix} 3 & 2 & 1 & 4 & 5 \\ 24 & 11 & 16 & 19 & 38 \\ -1 & -1 & -3 & 1 & -1 \\ -6 & 0 & -2 & -4 & -10 \end{bmatrix}$$

$$(f) AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, BA = \begin{bmatrix} -2 & -2i \\ -2i & 2 \end{bmatrix}.$$

$$4. (a) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & -\alpha/\beta^2 \end{bmatrix}$$

$$(d) \begin{bmatrix} 1/\alpha & 0 \\ -\beta/\alpha\gamma & 1/\gamma \end{bmatrix} \quad (e) \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}.$$

$$5. (a) A^2 = \begin{bmatrix} 10 & 3 & 5 \\ 3 & 3 & 2 \\ 3 & 4 & 8 \end{bmatrix}, A^3 = \begin{bmatrix} 25 & 21 & 33 \\ 9 & 10 & 12 \\ 27 & 6 & 13 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 124 & 59 & 96 \\ 45 & 26 & 37 \\ 66 & 53 & 87 \end{bmatrix}$$

$$(b) A^2 = \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix}, A^3 = \begin{bmatrix} 19 & 18 \\ 27 & 19 \end{bmatrix}, A^4 = \begin{bmatrix} 73 & 56 \\ 84 & 73 \end{bmatrix}$$

$$(c) A^2 = \begin{bmatrix} 10 & 5 & 10 \\ -5 & 4 & -1 \\ 5 & 45 & 50 \end{bmatrix}, A^3 = \begin{bmatrix} 35 & 60 & 80 \\ -20 & 3 & -12 \\ 20 & 340 & 355 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 125 & 520 & 595 \\ -75 & -54 & -104 \\ 75 & 2455 & 2505 \end{bmatrix}.$$

$$6. (a) \begin{bmatrix} \alpha & -2\beta \\ \beta & \alpha + \beta \end{bmatrix} \quad (b) \begin{bmatrix} \alpha & \beta \\ \beta & \alpha - \beta \end{bmatrix} \quad (c) \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$$

$$(d) \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}, \text{ where } \alpha \text{ and } \beta \text{ are arbitrary scalars.}$$

$$7. (a) \begin{bmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (b) \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{7}{2} & \frac{1}{2} \end{bmatrix} \quad (c) \text{ Solution does not exist.}$$

$$8. (a) \begin{bmatrix} \lambda_1 \mu_1 & 0 & \dots & 0 \\ 0 & \lambda_2 \mu_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \mu_n \end{bmatrix} \quad (b) \begin{bmatrix} \mu_1 \lambda_1 & 0 & \dots & 0 \\ 0 & \mu_2 \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_n \lambda_n \end{bmatrix}$$

$$(c) \begin{bmatrix} 1/\lambda_1 & 0 & \dots & 0 \\ 0 & 1/\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/\lambda_n \end{bmatrix}, \text{ if } \lambda_i \neq 0, i = 1, 2, \dots, n.$$

$$9. (a) \text{ Nilpotent; } 4 \quad (b) \text{ Nilpotent; } 4 \\ (c) \text{ Nilpotent; } 3 \quad (d) \text{ Not nilpotent.}$$

$$14. \begin{bmatrix} \alpha & 1 - \alpha^2 \\ \beta & -\alpha \end{bmatrix}, \begin{bmatrix} \alpha & \beta \\ 1 - \alpha^2 & -\alpha \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\pm \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ where } \alpha \text{ and } \beta \text{ are arbitrary scalars with } \beta \neq 0.$$

$$16. (a) \begin{bmatrix} -5 & 50 \\ -25 & 45 \end{bmatrix} \quad (b) \begin{bmatrix} -21 & 60 \\ -30 & 39 \end{bmatrix} \quad (c) \begin{bmatrix} 22 & -24 \\ -24 & 46 \end{bmatrix}$$

$$(d) \begin{bmatrix} -25 & 35 \\ 35 & -60 \end{bmatrix} \quad (e) \begin{bmatrix} 64 & -70 \\ -10 & 54 \end{bmatrix}$$

$$(f) \begin{bmatrix} 4119 & -2546 \\ -25460 & 37217 \end{bmatrix}.$$

18. (a)  $x = 1, y = 1$  (b)  $x = b/\beta, y = (a\beta - bx)/\beta^2, \beta \neq 0$

(c)  $x - y = 4z = 2.$

20.

	$I$	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$D_7$
$I$	$I$	$D_1$	$D_2$	$D_3$	$D_4$	$D_5$	$D_6$	$D_7$
$D_1$	$D_1$	$I$	$D_3$	$D_2$	$D_5$	$D_4$	$D_7$	$D_6$
$D_2$	$D_2$	$D_3$	$D_1$	$I$	$D_6$	$D_7$	$D_5$	$D_4$
$D_3$	$D_3$	$D_4$	$I$	$D_1$	$D_7$	$D_6$	$D_4$	$D_5$
$D_4$	$D_4$	$D_5$	$D_7$	$D_6$	$D_1$	$I$	$D_2$	$D_3$
$D_5$	$D_5$	$D_6$	$D_6$	$D_7$	$I$	$D_1$	$D_3$	$D_2$
$D_6$	$D_6$	$D_7$	$D_4$	$D_5$	$D_3$	$D_2$	$D_1$	$I$
$D_7$	$D_7$	$D_6$	$D_5$	$D_4$	$D_2$	$D_3$	$I$	$D_1$

### Problem Set 5.5

- (a) Range:  $\{(1, -1, 1), (3, 7, 0), (2, 2, 1)\} = V_3$ ; kernel:  $V_0$ ; rank: 3; nullity: 0

(b) Range:  $\{(1, 3), (-1, -2)\} = V_2$ ; kernel:  $\{(-1, 1, 1)\}$ ; rank: 2; nullity: 1

(c) Range:  $\{(2, 7, 3), (0, 1, -1), (1, 2, 1)\} = V_3$ ; kernel:  $V_0$ ; rank: 3; nullity: 0

(d) Range:  $\{(2, 0, 1, 2), (1, -1, 2, 0), (2, 2, 4, 3), (0, 1, 3, 0)\} = V_4$ ; kernel:  $\{(-15, 1, 9, 9, -12)\}$ ; rank: 4; nullity: 1

(e) Range:  $\{(1, 2, 1, 0), (-1, 3, 5, 0), (1, -1, 2, 1), (0, 1, 0, 1)\} = V_4$ ; kernel:  $V_0$ ; rank: 4; nullity: 0

(f) Range:  $\{(-1, 3, 2), (1, 1, 2), (1, -1, 1)\} = V_3$ ; kernel:  $V_0$ ; rank: 3; nullity: 0.

$$2. (a) \frac{1}{2} \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \quad (b) \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 5 & -4 & 1 \end{bmatrix}$$

$$(c) \frac{1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 3 & -1 & 5 \\ 3 & -1 & 3 \end{bmatrix}$$

$$(d) \frac{1}{24} \begin{bmatrix} 7 & -13 & 8 & 9 \\ -3 & 9 & 0 & 3 \\ 4 & -4 & 8 & -12 \\ 2 & 10 & -8 & 6 \end{bmatrix}$$

$$(e) \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$3. \alpha^3 + \beta^3 \neq 0; \frac{1}{\alpha^3 + \beta^3} \begin{bmatrix} \alpha^2 & -\alpha\beta & \beta^2 \\ \beta^2 & \alpha^2 & -\alpha\beta \\ -\alpha\beta & \beta^2 & \alpha^2 \end{bmatrix}.$$

### Problem Set 5.6

$$2. (a) \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 1 \\ 3 & 2 & -2 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ -1 & 2 \end{bmatrix} \quad (c) \begin{bmatrix} -2 & 7 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$$

$$(d) \begin{bmatrix} 4 & -2 & 7 \\ -1 & 1 & -3 \\ 1 & -3 & 8 \end{bmatrix} \quad (e) \begin{bmatrix} -3 & -4 & 8 \\ 8 & 4 & -4 \end{bmatrix}$$

$$(f) \begin{bmatrix} -2 & -3 & 5 \\ 7 & 0 & 4 \end{bmatrix} \quad (g) \begin{bmatrix} 3 & -2 \\ -2 & 10 \end{bmatrix} \quad (h) \begin{bmatrix} 2 & 2 & 2 \\ 2 & -1 & 4 \end{bmatrix}$$

$$(i) \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}.$$

$$5. (a) \begin{bmatrix} 1-i & -1+i \\ 3+i & 2+i \end{bmatrix} \quad (b) \begin{bmatrix} 1+i & 2-i \\ 2+3i & 1-4i \\ 1-i & 3+2i \end{bmatrix}$$

$$(c) \begin{bmatrix} 2-3i & 1+2i & 3-4i \\ 2+i & 2+2i & 2-i \end{bmatrix}$$

$$(d) \begin{bmatrix} 1 & 2+i & 1-i \\ 1+i & -i & 2-i \\ -1-2i & i & 3-2i \end{bmatrix}.$$

6.  $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$ .

9. (b) A diagonal matrix.

### Problem Set 5.7

$$1. (a) \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(c) \begin{bmatrix} 1 & 0 & 1 & 0 & 3 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (d) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(e) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (f) \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \quad (g) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(h) \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (i) \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2. (a) 2 (b) 2 (c) 3 (d) 4 (e) 4 (f) 3 (g) 3  
(h) 3 (i) 3.

3. Same as the answer to Problem 2.

$$4. (d) \text{ Nonsingular; } \frac{1}{7} \begin{bmatrix} 1 & -5 & 1 & 5 \\ 4 & 1 & -3 & -1 \\ -3 & 1 & 4 & -1 \\ 1 & 16 & -6 & -9 \end{bmatrix}$$

(e) Nonsingular;  $\frac{1}{3} \begin{bmatrix} 3 & 3 & -3 & 3 \\ 7 & 3 & -5 & 4 \\ -1 & 0 & 2 & -1 \\ 5 & 3 & -4 & 2 \end{bmatrix}.$

5. (a)  $x = 14/19, y = 3/19, z = 13/19$   
 (b)  $x = 7/4, y = 1/2, z = -5/4$   
 (c)  $x = 1/9, y = -2/9, z = -1/3$   
 (d)  $x = 5/7, y = 1, z = 3/7$   
 (e)  $x = 8/5, y = 5, z = 9/5.$
7.  $k_1 = 2, k_2 = 4, k_3 = 6, k_4 = 8, k_5 = 9, k_6 = 10.$

### Problem Set 5.8

1. (a) Consistent;  $x_1 = 5/16, x_2 = 1/16, x_3 = 3/16, x_4 = 1/8$   
 (b) Consistent;  $x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 0$   
 (c) Consistent;  $(1, 0, 0, 0) + [(0, -4, 1, 1, 0), (0, 3, -1, 0, 1)]$   
 (d) Consistent;  $(1, 0, 1, 0) + [(-2, 1, 0, 0), (0, 0, -2, 1)]$   
 (e) Not consistent (f) Not consistent  
 (g) Consistent;  $\frac{1}{3}(2, 1, 1, 0) + [(2, -8, -5, 9)]$   
 (h) Consistent;  $x_1 = 7/6, x_2 = 41/6, x_3 = 1/3, x_4 = -59/6$   
 (i) Not consistent (j) Not consistent  
 (k) Consistent;  $x_1 = x_2 = 2, x_3 = 0$   
 (l) Consistent;  $x_1 = -x_2 = 1, x_3 = 0$   
 (m) Consistent;  $(7, 6, 14, 3, 0) + [(0, 14, 14, 1, 1), (7, -1, 12, 3, 4)]$   
 (n) Consistent;  $(1, -2, 0, 0) + [(-4, 7, 26, 1)].$

### Problem Set 5.9

1. (a) Nonsingular;  $\frac{1}{3} \begin{bmatrix} 7 & -3 & -8 \\ 3 & -1 & -4 \\ -7 & 3 & 10 \end{bmatrix}$

(c) Nonsingular;  $\frac{1}{4} \begin{bmatrix} -3 & 1 & 1 \\ 1 & 1 & -3 \\ 10 & -2 & -2 \end{bmatrix}$



$$(e) \text{ Nonsingular; } \frac{1}{25} \begin{bmatrix} 16 & 7 & -1 & -7 \\ -6 & 1 & 4 & -1 \\ 7 & -6 & 5 & 6 \\ -7 & 6 & -5 & 23 \end{bmatrix}$$

$$(f) \text{ Nonsingular; } \frac{1}{4} \begin{bmatrix} -3 & -1 & 2 \\ 5 & 3 & -2 \\ -4 & -4 & 4 \end{bmatrix}$$

2. Same as the answer to Problem 2 of Problem Set 5.5.  
 3. Same as the answer to Problem 4 of Problem Set 5.7.

### Problem Set 6.1

1. (a) Even (b) Odd (c) Odd (d) Odd.  
 2. (a) Even : (1, 2, 3), (2, 3, 1), (3, 1, 2)  
     Odd : (1, 3, 2), (2, 1, 3), (3, 2, 1)  
     (b) Even : (1, 2, 3, 4), (1, 3, 4, 2), (1, 4, 2, 3), (2, 1, 4, 3),  
         (2, 3, 1, 4), (2, 4, 3, 1), (3, 2, 4, 1), (3, 1, 2, 4),  
         (3, 4, 1, 2), (4, 2, 1, 3), (4, 3, 2, 1), (4, 1, 3, 2)  
     Odd : (1, 2, 4, 3), (1, 3, 2, 4), (1, 4, 3, 2), (2, 1, 3, 4),  
         (2, 3, 4, 1), (2, 4, 1, 3), (3, 2, 1, 4), (3, 1, 4, 2),  
         (3, 4, 2, 1), (4, 2, 3, 1), (4, 3, 1, 2), (4, 1, 2, 3).  
 3. (a) 6 (b) 10.  
 4. (a)  $(x - y)(y - z)(z - x)$  (b)  $(x - y)(y - z)(z - x)$   
     (c)  $a^3 + b^3 + c^3 - 3abc$ .

### Problem Set 6.4

1.  $n^2$ .  
 3. (a) -20 (b) -4 (c) 6 (d) 6 (e) 500 (f) 0.  
 6. 
$$\begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_n & x_1 & x_2 & \dots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \dots & x_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ x_2 & x_3 & x_4 & \dots & x_1 \end{vmatrix} = \prod_{i=1}^n (x_1 + x_2\omega_i + x_3\omega_i^2 + \dots + x_n\omega_i^{n-1}),$$

where  $\omega_1, \omega_2, \dots, \omega_n$  are  $n$ -th roots of unity.

$$7. (b) \begin{bmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{bmatrix} = 0.$$

$$8. x = 0, -a - b - c.$$

$$10. (a) \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 2 & 0 & 0 \\ -4 & 2 & 0 \\ 5 & -4 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} -1 & 1 & -1 \\ 3 & -1 & 5 \\ 3 & -1 & 3 \end{bmatrix}$$

$$(d) \begin{bmatrix} -7 & 13 & -8 & -9 \\ 3 & -9 & 0 & -3 \\ -4 & 4 & -8 & 12 \\ -2 & -10 & 8 & -6 \end{bmatrix} \quad (e) \begin{bmatrix} -2 & 2 & 0 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

### Problem Set 6.5

- Same as the answer to Problem 1 of Problem Set 5.5.
- Same as the answers to Problems 1 and 2 of Problem Set 3.5.
- Same as the answer to Problem 2 of Problem Set 5.5.

### Problem Set 6.6

- 0.

### Problem Set 6.7

- $x = -1, y = 4$
  - $x = 31/11, y = -5/11$
  - $x = -2, y = 29/2, z = -8$
  - $x = -1/2, y = 5/2, z = 2$
  - $x = 25/2, y = -17/2, z = -1/2$ .
- $x_1 = 5/16, x_2 = 1/16, x_3 = 3/16, x_4 = 1/8$
  - $x_1 = 2x_2 = 2, x_3 = x_4 = 0$
  - $x_1 = 7/6, x_2 = 41/6, x_3 = 1/3, x_4 = -59/6$ .
- Inverses of the coefficient matrices :

$$(a) \begin{bmatrix} 2 & -1 \\ -5 & 3 \end{bmatrix} \quad (b) \frac{1}{11} \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \quad (c) \begin{bmatrix} -1 & -1 & 1 \\ 4 & 5 & -\frac{7}{2} \\ -2 & -3 & 2 \end{bmatrix}$$

$$(d) \begin{bmatrix} -1 & \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} & 1 \\ 1 & -2 & 1 \end{bmatrix} \quad (e) \begin{bmatrix} -1 & \frac{1}{2} & -1 \\ 0 & -\frac{1}{2} & 1 \\ 1 & -\frac{1}{2} & 0 \end{bmatrix}.$$

The solutions are the same as those of Problem 1.

4.  $x' = x \cos \alpha + y \sin \alpha$ ,  $y' = -x \sin \alpha + y \cos \alpha$ .

### Problem Set 6.8

- $0, 5$ ;  $K_0 = [(1, -3)]$ ,  $K_5 = [(1, 2)]$
  - $i, -i$ ;  $K_i = [(i, 1)]$ ,  $K_{-i} = [(-i, 1)]$
  - $-3, 2$ ;  $K_{-3} = [(1, -1)]$ ,  $K_2 = [(3, 2)]$
  - $-1, -1, 8$ ;  $K_{-1} = [(1, -2, 0), (0, -2, 1)]$ ,  $K_8 = [(2, 1, 2)]$
  - $1, 1, 0$ ;  $K_1 = [(0, 1, 2)]$ ,  $K_0 = [(0, 0, 1)]$
  - $-i, -i, 2i$ ;  $K_{-i} = [(1, 0, -1), (0, 1, -1)]$ ,  $K_{2i} = [(1, 1, 1)]$
  - $-1, \frac{1}{2}, 1, 3$ ;  $K_{-1} = [(1, 0, -4, 0)]$ ,  $K_{1/2} = [(-8, 6, 8, -3)]$ ,  $K_1 = [(-3, 2, 2, 0)]$ ,  $K_3 = [(1, 0, 0, 0)]$ .

$$2. \quad 1(b) \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad 1(c) \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$1(g) \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

$$3. \quad \lambda^4 - 1; \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

### Problem Set 6.9

- $Ce^{4x}$
  - $C$
  - $-4 / |1 - x^2|$
  - $-2 |x|^3$ , where  $C$  is an arbitrary constant.
- 0.

### Problem Set 6.10

- $4i + j - 5k$
  - $4i - 7j + 2k$
  - $(-3, 1, -7)$

(d)  $(1, 5, -3)$ .

4.  $u \times v = 0$ .

5. (a) No (b) Yes (c) Yes.

6. (a)  $8x + 4y - z = 19$  (b)  $y - z + 1 = 0$  (c)  $8x - 7z = 11$ .

8.  $(a \cdot b \times d)c - (a \cdot b \times c)d$  or  $(a \cdot c \times d)b - (b \cdot c \times d)a$ .

9.  $11/\sqrt{26}$ .

10.  $3x - 4y + z = 11$ .

11.  $x/11 = y/2 = (3 - 2z)/14$ .

12.  $4x + y + 11z = 39$ .

13.  $3x + 4y + 7z = 13$ .

**Problem Set 7.2**

1. 1(a)  $\{(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}), (-\frac{7}{\sqrt{66}}, \frac{4}{\sqrt{66}}, -\frac{1}{\sqrt{66}}),$   
 $(-\frac{1}{\sqrt{11}}, -\frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}})\}$

1(b)  $\{(1, 0, 0), (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\}$

1(d)  $\{(\frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}), (-\frac{1}{\sqrt{195}}, -\frac{5}{\sqrt{195}}, \frac{13}{\sqrt{195}}),$   
 $(\frac{5}{\sqrt{26}}, -\frac{1}{\sqrt{26}}, 0)\}$ .

2. 2(b)  $\{(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0), (\frac{1}{\sqrt{30}}, -\frac{5}{\sqrt{30}}, \frac{2}{\sqrt{30}}, 0),$   
 $(\frac{12}{\sqrt{205}}, 0, -\frac{6}{\sqrt{205}}, \frac{5}{\sqrt{205}}), (\frac{2}{\sqrt{41}}, 0, -\frac{1}{\sqrt{41}}, -\frac{6}{\sqrt{41}})\}$

2(d)  $\{(0, 0, 1, 0), (\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}),$   
 $(-\frac{17}{\sqrt{14286}}, -\frac{46}{\sqrt{14286}}, 0, \frac{109}{\sqrt{14286}})\}$ .

3. 3(a)  $\{\frac{\sqrt{15}}{4}x^2 - \frac{\sqrt{15}}{4}, \frac{5}{4}x^2 + x - \frac{1}{4},$   
 $-\frac{5}{2\sqrt{2}}x^2 + \frac{1}{\sqrt{2}}x + \frac{1}{2\sqrt{2}}\}$

3(c)  $\{\frac{\sqrt{3}}{\sqrt{2}}x, \frac{5\sqrt{7}}{2\sqrt{2}}x^2 - \frac{3\sqrt{7}}{2\sqrt{2}}x, \frac{3\sqrt{35}}{2\sqrt{94}}(x^4 + x^2),$   
 $\frac{\sqrt{5}}{\sqrt{11362}}(47 - 42(x^4 + x^2))\}$ .

5. (a) 0 (b) 0.

8. If  $v_1$  is the first vector in the span of the earlier ones, then the corresponding vector in the process of orthogonalisation will be zero.

### Problem Set 7.3

6. Yes.

### Problem Set 7.4

$$1. \quad (a) \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (b) \begin{bmatrix} 18 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad (c) \begin{bmatrix} 7 & 0 & 0 \\ 0 & \sqrt{7} & 0 \\ 0 & 0 & -\sqrt{7} \end{bmatrix}.$$

$$2. \quad (a) \quad 4x^2 - 9y^2 = 36 \quad \text{or} \quad -9x^2 + 4y^2 = 36$$

$$(b) \quad 9x^2 - 4y^2 = 36 \quad \text{or} \quad -4x^2 + 9y^2 = 36$$

$$(c) \quad 9x^2 + 4y^2 = 36 \quad \text{or} \quad 4x^2 + 9y^2 = 36.$$

### Problem Set A1

$$1. \quad (a) \quad y = C_1 e^{-x} + C_2 x e^{-x} \quad (b) \quad y = C_1 \cos 2x + C_2 \sin 2x$$

$$(c) \quad y = C_1 + C_2 \cos x + C_3 \sin x \quad (d) \quad y = C_1 e^{2x/3} + C_2 e^{-x}$$

$$(e) \quad y = C_1 e^x + e^{-x/2} (C_2 \cos(\sqrt{3}x/2) + C_3 \sin(\sqrt{3}x/2))$$

$$(f) \quad y = C_1 e^{3x} + C_2 e^{-x} \quad (g) \quad y = C_1 e^{-(1+\sqrt{2})x} + C_2 e^{-(1-\sqrt{2})x}$$

$$(h) \quad y = C_1 e^x + C_2 x e^x + C_3 e^{-x}$$

$$(i) \quad y = C_1 + C_2 x + C_3 e^x + C_4 e^{-5x}$$

$$(j) \quad y = C_1 + C_2 e^{3x} + C_3 e^{-2x}$$

$$(k) \quad y = ((C_1 + C_2 x) \cos(\sqrt{7}x) + (C_3 + C_4 x) \sin(\sqrt{7}x)) e^{-2x}$$

$$(l) \quad y = (C_1 + C_2 x) e^x + (C_3 + C_4 x) e^{-x}$$

$$(m) \quad y = (C_1 + C_2 x) e^{2x} + (C_3 + C_4 x) e^x$$

$$(n) \quad y = C_1 e^{-(3+\sqrt{5})x/2} + C_2 e^{-(3-\sqrt{5})x/2}$$

$$(o) \quad y = C_1 e^{x/2} + C_2 e^{-3x} + C_3 \quad (p) \quad y = C_1 + (C_2 + C_3 x) e^{2x}.$$

### Problem Set A2

$$1. \quad (a) \quad y = C_1 \cos 2x + C_2 \sin 2x - (x^2/8) \cos 2x + (x/16) \sin 2x$$

$$(b) \quad y = C_1 e^{2x} + C_2 x e^{2x} + (2x + 1) e^{-2x/32}$$

$$(c) \quad y = C_1 \cos 2x + C_2 \sin 2x + (1/4) \cos 2x \ln \cos 2x \\ + (x/2) \sin 2x \quad (d) \quad y = C_1 e^{2x/3} + C_2 e^{-x} + 3e^{x/2}$$

$$(e) \quad y = C_1 \cos(x/2) + C_2 \sin(x/2) + \sin(x/2) \ln(\sec(x/2) \\ + \tan(x/2)) - 1 \quad (f) \quad y = C_1 \cos x + C_2 \sin x - (x/2) \cos x$$

- (g)  $y = C_1 e^{-2x} + C_2 x e^{-2x} + x^2 e^{-2x}/6$   
 (h)  $y = C_1 e^{x/2} + C_2 e^{-x/3} + (3x + 11)e^{-x}/9$   
 (i)  $y = C_1 + C_2 \cos x + C_3 \sin x - 2x + x^3/3$   
 (j)  $y = C_1 + C_2 e^x + C_3 e^{-x} - \cos x - (x/2) \sin x$   
 (k)  $y = C_1 e^{3x} + C_2 e^x + C_3 e^{-x} + x e^{3x}/8$   
 (l)  $y = C_1 + C_2 e^{x/2} + C_3 e^{-x} + 2x e^{x/2}/3.$

### Problem Set A3

1. (a)  $y = C_1 e^{3x} + C_2 e^{-x} - e^{2x}/3$   
 (b)  $y = C_1 \cos 2x + C_2 \sin 2x - (x/4) \cos 2x$   
 (c)  $y = (C_1 \cos x + C_2 \sin x)e^{-x} + \frac{x}{5} e^x - \frac{4}{25} e^x$   

$$+ \frac{1}{2} x^2 - x + \frac{1}{2}$$
  
 (d)  $y = C_1 e^{(-1 + \sqrt{2})x} + C_2 e^{(-1 - \sqrt{2})x} + \sinh x$   
 (e)  $y = C_1 e^x + C_2 x e^x + C_3 e^{-x} + (2 \cos x + 2 \sin x + 2x e^{-x} + x^2 e^{-x} + 8)/8$   
 (f)  $y = C_1 + C_2 \cos 3x + C_3 \sin 3x + x^2/18$   
 (g)  $y = (C_1 \cos(\sqrt{3}x/2) + C_2 \sin(\sqrt{3}x/2))e^{-x/2} + C_3 e^x + C_4 e^{-x}$   

$$x^2 + 2x - 5 + \frac{1}{12}((3x + 2x^2) \cosh x - (6x + x^2) \sinh x)$$
  
 (h)  $y = C_1 + C_2 \cos x + C_3 \sin x - x \cos x$   
 (i)  $y = C_1 + C_2 x + C_3 e^x + C_4 e^{-x} - \frac{2}{25} x^2 - \frac{x^3}{30} + \frac{107}{338} \cos x$   

$$- \frac{27}{169} \sin x + \frac{1}{13} x \cos 2x + \frac{3}{26} x \sin x$$
  
 (j)  $y = C_1 + C_2 e^{2x} + C_3 e^{-2x} - x^2/12 + x/36 + \frac{7}{50} \cos x$   

$$+ \frac{1}{50} \sin x - \frac{1}{18} e^{-3x} + \frac{7}{100} x e^{-2x} + \frac{1}{20} x^2 e^{-2x}$$
  
 (k)  $y = (C_1 \cos(\sqrt{7}x) + C_2 \sin(\sqrt{7}x))e^{-2x} + \frac{1}{7} x e^{-2x}$   

$$- \frac{1}{2} e^{-2x} \sin 3x.$$

### Problem Set A4

1. (a)  $\frac{1}{4} (2x - 1)e^x$  (b)  $\frac{1}{4} (2x + 3 + 2e^{3x})$

$$(c) -x^3/10 - 4x/25 - \frac{1}{13} \sin x + \frac{3}{26} \cos x$$

$$(d) \frac{1}{9}(3x + 2)e^{-x} - \frac{1}{9}(3x - 2).$$

$$2. (a) -e^{-2x}/29 \quad (b) \frac{1}{108}(\sin 3x - 6x \cos 3x)$$

$$(c) \frac{1}{101}(\sin 2x - 10 \cos 2x)e^x$$

$$(d) -\frac{1}{20}(2 \cos 2x + \sin 2x)$$

$$(e) \frac{1}{1250}(250x^3 + 475x^2 - 1940x + 388)e^x$$

$$(f) -e^{2x} \cos x \quad (g) \frac{1}{147}(7x + 2)e^{-3x} \quad (h) \frac{1}{840}x^7e^{2x}.$$

# Index

- Abel's formula, 240
- addition function, 12
- additive inverse, 42
- associative operation, 19, 20, 134, 167, 291
- auxiliary equation, 277
  
- Base space, 81
- basis, 95
  - orthonormal, 262
  - standard, 97, 161
- Bernoulli's equation, 143
- binary operation, 17
  
- Cartesian product, 6
- central quadric, 267
- characteristic
  - equation, 234
  - polynomial, 234
  - value, 234
  - vector, 234
- circulant, 223
- coefficient matrix, 195
- cofactor, 216
- collinear vectors, 89
- column, 150
  - rank, 187
  - vector, 148
- commutative operation, 18, 20, 291
- complement, 5
- complementary function, 142, 276
- composition, 24, 133
- coordinate vector, 99
- coplanar vectors, 89
- Cramer's rule, 231
- cross product, 241
  - direction of, 242
  - properties of, 242
  
- DeMorgan's theorem, 7
- derivative family, 285
  - modified, 286
- determinant, 204
  - expansion of, 204, 219
  - minor, 224
  - product of, 227, 228
- diagonalisation, 237, 270, 271
- differential equation, ordinary li-  
near, 141, 144, 276
  - homogeneous, 276
  - nonhomogeneous, 280
- differential operator, 109
- dimension, 96, 100, 101
- direct sum, 80
- directed line segment, 37
- direction cosines, 59
- distance between points, 36
- distributive operation, 21, 134, 167, 291
- domain, 10
- dot product, 50
  - properties of, 50
  
- Echelon form, 188
- eigenspace, 236
- eigenvalues, 234, 236



- eigenvectors, 234, 236
- elementary row operation, 187
- equality in  $V_n$ , 64
- equivalent system, 185
- Euclidean space, 258
- existence and uniqueness theorem, 144, 239
- extension of a basis, 99
- Field, 20
- finite-dimensional, 96
- function(s), 10
  - absolute value, 12
  - addition of, 25
  - constant, 12
  - equal, 24
  - linear, 31
  - one-one, 14
  - onto, 13
  - pointwise sum of, 25
  - polynomial, 12
  - real-valued, 25
  - scalar multiplication of, 26
  - zero, 12
- Gram-Schmidt orthogonalisation, 261, 274
- group, 21
  - abelian, 21
  - commutative, 21
  - orthogonal, 265
  - unitary, 265
- Homogeneous equation, 139
- hyperboloid of one sheet, 273
- Identity, 21
  - function, 12
  - map, 109, 115, 122
- image, 10
- indicial equation, 277
- infinite-dimensional, 96
- initial value problem, 143
- inner product, 50, 164, 258
  - natural, 259
  - space, 258
- intersection, 5
  - of subspaces, 77
- inverse
  - of element, 21
  - of function, 29
  - of linear transformation, 122, 123, 125
  - of matrix, 167, 177, 200, 226
- inversion, 203
  - matrix, 200
- invertible, 168
- isomorphic, 127
- isomorphism, 122, 127, 160
- Kernel of a linear transformation, 114, 116, 139
  - matrix, 176
- Kronecker delta, 152
- Leader, 81
- left-handed system, 34
- line
  - equation of, 56
  - in vector space, 89
- linear
  - combination, 74
  - finite, 74
  - nontrivial, 86
  - trivial, 86
  - equation, system of, 195
  - extension, 111
  - operator, 107
  - transformation, 107
    - idempotent, 137
    - natural matrix of, 156
    - nilpotent, 137
    - nonsingular, 122, 127, 135
    - nullity of, 117
    - product of, 134
    - range of, 114, 116
    - rank of, 117
    - scalar multiple of, 129

- similar, 253
  - sum of, 129
  - variety, 81, 139
- linearly
  - dependent, 87, 88, 93
  - independent, 87, 88, 93
- Map, 10
  - linear, 107, 155
  - zero, 109, 115
- matrix, 149, 155
  - addition of, 159
  - adjoint, 222
  - augmented, 195
  - characteristic equation of, 234
  - characteristic polynomial of, 234
  - characteristic value of, 234
  - characteristic vector of, 234
  - conjugate, 182
  - diagonal, 181
  - diagonalisation of, 237, 270, 271
  - eigenvalue of, 234, 236
  - eigenvector of, 234, 236
  - entries of, 150
  - equal, 155
  - Hermitian, 182
  - idempotent, 182, 229
  - identity, 152
  - involutory, 172, 230
  - lower triangular, 182
  - multiplication of, 164
  - nilpotent, 171, 229
  - nonsingular, 168
  - nullity of, 176
  - orthogonal, 264
  - polynomial, 172
  - product of, 164, 165
  - proper value of, 234
  - proper vector of, 234
  - range of, 175
  - rank of, 175, 188
  - scalar, 182
  - scalar multiple of, 159
  - similar, 253
  - singular, 168
  - skew-Hermitian, 182
  - skew-symmetric, 182
  - square, 155
  - sub-, 224
  - sum of, 159
  - symmetric, 182
  - transpose of, 180
  - transposed conjugate of, 182
  - unitary, 264
  - upper triangular, 182
  - zero, 153
- minor, 224
- multiplication function, 12
- Natural matrix, 156
- negative triple, 243
- nilpotence, degree of, 137, 171
- nonhomogeneous equation, 281, 139
- nontrivial solution, 139
- norm, 259
- normal, 141
- $n$ -tuples. 64
  - addition of, 64
- null set, 4
- null space, 114
- nullity, 117, 176
- One-one correspondence, 15
- onto. 13
- operational methods, 289
- operator equation, 138
  - polynomial, 290, 292
- order, 150
- orthogonal set, 261
- orthogonalisation, Gram-Schmidt, 261
- orthogonally similar, 266
- Parallel, 81
- particular integral, 140, 280
  - solution, 140, 276
- permutation
  - even, 203

- odd, 203
- perpendicular distance, 62
- plane
  - equation of, 60, 245
  - in a vector space, 89
  - vectors, 40
    - addition of, 41
- pointwise product, 27
  - scalar multiplication, 26
  - sum, 25
- polynomial, degree of, 12
- position vector, 40
- positive triple, 242
- pre-image, 13, 141
- proper
  - values, 234
  - vectors, 234
- Quadric, 267
  - tangent to, 267
- Range, 13, 114, 116, 175
- rank, 117, 175, 188
  - column, 187
  - row, 187
- rank-nullity theorem, 120
- reflection, 109
- relation, 8
  - equivalence, 8, 128, 194
  - reflexive, 8
  - symmetric, 8
  - transitive, 8
- right-handed system, 34
- ring, 23
  - commutative, 23
  - with identity, 23
- row, 150
  - equivalent, 187
  - operations, 187, 207, 209
  - rank, 187
  - reduced echelon form, 188
  - reduction, 188
  - vector, 148
- Scalar, 63
  - multiple, 26, 39, 42, 64, 129, 159
  - multiplication, function, 12
  - multiplication, properties of, 29, 43
  - product, 50
  - projection, 53
  - triple product, 244
    - properties of, 244
- Schwarz inequality, 51, 260
- sequence, 13
- set(s), 1
  - addition of, 78
  - empty, 4
  - equal, 3
  - finite, 15
  - infinite, 6
  - null, 4
  - of complex valued functions, 66
  - of continuous functions, 66
  - of infinite sequences, 124
  - of linear transformations, 130
  - of  $m \times n$  matrices, 156
  - of  $n$ -times differentiable functions, 66
  - of  $n$ -tuples of complex numbers, 68
  - of  $n$ -tuples of real numbers, 64
  - of plane vectors, 41
  - of polynomials with complex coefficients, 66
  - of polynomials with real coefficients, 12, 66
  - of real-valued functions, 25, 66
  - of space vectors, 41
- shortest distance, 246
- singleton, 4
- skew lines, 247
- space
  - coordinates, 33
  - vectors, 40
    - addition of, 41
- span, 89

- stairs, 188
- steps, 188
- straight line
  - in a plane, 57
  - in space, 58
  - vector equation of, 57
- submatrix, 224
- subset, 3
  - proper, 4
  - span of, 74
- subspace, 69
  - nontrivial, 72
  - trivial, 72
- superset, 3
- symmetric form, 57, 58, 61
  
- Target set, 10
- translate, 81
- translation function, 12, 109
- triangle inequality, 260
- trivial solution, 139, 196
  
- Undetermined coefficient, method of, 285
- union, 5
- uniqueness, existence and, 144, 195
- unitarily similar, 266
- unitary space, 258
  
- Variation of parameters, 280
- vector(s), 37, 64
  - angle between, 48
  - column, 148
  - components of, 46
  - coordinates of, 99
  - difference of, 44
  - direction of, 41
  - equal, 41
  - free, 38
  - initial point of, 33, 37
  - length of, 37, 40
  - magnitude of, 37, 40
  - norm of, 259
  - orthogonal, 52, 261
  - parallel, 37
  - plane, 40
  - product, 241
  - projection, 53, 261
  - resolution of, 52
  - row, 148
  - scalar multiple of, 39, 42
  - sum of, 39
  - terminal point of, 33, 37
  - unit, 37, 44
- vector space, 62
  - complex, 62
  - finite-dimensional, 96
  - real, 62
  
- Wronskian, 238
  
- Zero vector
  - in plane, 40
  - in space, 40
  - of vector space, 64

## Errata

f.t. = from top  
f.b. = from bottom

Page	Line	In place of	Read as
4	6 f.b.	page 3	page 4
5	1 f.t.	$zz \mid 2z \vdash 2z \quad 5 = 0$	$zz \vdash 2z \vdash 2z \quad 5 = 0$
8	6 f.b.	... because $x \sim y$ ...	... because $x \sim x$ ...
11	1 f.b.	$B - ( \dots )$	... $B - \{ \dots \}$
12	6 f.b.	s called ...	is called ...
14	5 f.t.	... $f: x \rightarrow x^2$ ...	... $f: x \mapsto x^2$ ...
16	8 f.b.	... Problems 4 and 5.	... Problems 3 and 4.
16	7, 6 f.b.	... Problem 4 are one-one and which onto.	... Problem 3 are one-one and which are onto.
18	13 f.t.	... elements $(x, y)$ ...	... elements $x, y$ ...
20	7 f.t.	... real numbers.	... integers.
20	8 f.t.	... $Z \times Z$ ...	... $Z \times Z_0$ ...
22	9 f.t.	In all these four examples of groups the operation is commutative.	In all the examples of groups listed above the operation is commutative.
25	9 f.t.	... $(f \circ g)(x) = f(g(x)) = f(x^2)$	... $(f \circ g)(x) = f(g(x)) = f(x^2)$
25	19 f.t.	... by $\mathcal{F}(AR)$ .	... by $\mathcal{F}(A, R)$ .
26	1 f.t.	... function $x \mapsto x^2 + x + 1$	... function $x \mapsto x^2 + x + 1$
26	2 f.b.	... function 1/2 sine ...	... function (1/2) sine ...

Page	Line	In place of	Read as
20	6 f.t.	$\dots f^{-1}(y) = y^{-1} \dots$	$\dots f^{-1}(y) = y - 1$
44	13 f.t.	$\dots$ by $u$ and $v$ .	$\dots$ by $u$ and $-v$ .
48	1 f.b.	$\dots 0 \leq \theta < \pi \dots$	$\dots 0 < \theta < \pi \dots$
49	4 f.b.	$\dots (a_1, {}_2a) \dots$	$\dots (a_1, a_2) \dots$
51	20 f.t.	$\dots$ (or $V$ ) <sub>2</sub> $\dots$	$\dots$ (or $V_2$ ) $\dots$
54	19 f.t.	$\dots$ follow	$\dots$ follows
83	3 f.b.	$\dots \in V^2 \dots$	$\dots \in V_2 \dots$
85	8 f.t.	$\dots$ in $(\gamma', 0, 2)$ .	$\dots$ in $\gamma'(0, 2)$ .
92	3 f.b.	$1(1, 1, 0), -1(0, 1, 1) \dots$	$1(1, 1, 0) - 1(0, 1, 1) \dots$
96	10 f.t.	$\dots$ because $\{v_1, \dots, v_n\} \dots$	$\dots$ because $\{v_1, \dots, v_n\} \dots$
97	10 f.b.	Take $V \in V$ .	Take $v \in V$ .
97	8 f.b.	$v_2, v_3, \dots, v_r$ , say $v$ , $\dots$	$v_2, v_3, \dots, v_n, v \dots$
99	2 f.b.	$\dots$ exists any nonzero $\dots$	$\dots$ exists a nonzero $\dots$
100	1 f.t.	$\dots$ is L; otherwise $\dots$	$\dots$ is I; because otherwise $\dots$
102	11 f.t.	The reverse in equality $\dots$	The reverse inclusion $\dots$
102	22 f.t.	$\dim V_3 = (xy\text{-plane})$ $+ \dots$	$\dim V_3 = \dim$ $(xy\text{-plane}) + \dots$
120	1 f.b.	$T(\alpha_{n+1}u_{n+1} + \dots + 1$ $+ \alpha_p u_p) = 0$	$T(\gamma_{n+1}u_{n+1} + \dots + \alpha_p u_p)$ $= 0$
126	3 f.t.	$(y_1 + y_3)e_1 + \dots$	$(y_1 + y_3)e_1 + \dots$
126	2nd row in Table 4.2	$e_1 \dots e$	$e_1 - e_2$
133	7, 8, 11, 28 f.t.	$S \circ T$	$(S \circ T)$
151	last entry of 2nd matrix	$-\frac{3}{4}$	$-\frac{3}{4}$
167	3 f.t.	$BA = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$	$BA = \begin{bmatrix} 3 & 0 \\ -1 & -1 \end{bmatrix}$

Page	Line	In place of	Read as
174	22 f.t.	$\dots + x^2 \dots$	$\dots + x_2 \dots$
181	4 f.t.	$\dots \text{ and } = \beta'_{ij} = \beta_{ji} \dots$	$\dots \text{ and } \beta'_{ij} = \beta_{ji} \dots$
182	3 f.b.	$\dots (\bar{A})^T = (A^T).$	$\dots (\bar{A})^T = (\bar{A}^T)^{\sim}$
187, 188		The $7 \times 11$ matrix on page 187 should be read after line 9 f.t. of page 188.	
190	1 f.b.	$\alpha_{21} \quad \alpha_{32} \quad \alpha_{33} \quad \alpha_{34}$	$\alpha_{31} \quad \alpha_{32} \quad \alpha_{33} \quad \alpha_{34}$
251	13 f.t.	$= A(u_j =$	$= A(u_j) =$
267	12 f.b.	$u \, A u = k,$	$u^T A u = k,$















